Fixed-point exam. March, 24, 2022

Teacher : Philippe Bich. Durée : 2H from 15H30 to 17H30. No documents, no electronic devices

Exercise 1

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$f(x,y) = (\frac{e^x}{1+x^2} - 6y, x + cos(xy))$$

Prove there exists at least one element $(x,y) \in \mathbf{R}^2$ such that $f(x,y) = (0,0)$.
(Hint : you can use the mapping $g(x,y) = (-6y, x)$.)

Exercise 2 a) Let $X = \{(x, \sqrt{x}) : x \in [0,1]\}$. Is X convex ? Let $f : X \to X$ continuous. Does there exist $(x,y) \in X$ such that $f(x,y) = (x,y)$? (prove the general case if yes, find a counterexample if no). b) Let $X = \{(x, \sqrt{x}) : x \in [0, +\infty[\}$. Let $f : X \to X$ continuous. Does there always exist $(x,y) \in X$ such that $f(x,y) = (x,y)$? (prove the general case if yes, find a counterexample if no).

Exercise 3 Let $X = \{(x,y) \in [-1,1] \times [-1,0] : x^2 + y^2 \leq 1\} \cup \{0\} \times [0,1]$. Represent X graphically (union of a half disck and a segment). Let $f : X \to X$ continuous. Prove there exists $(x,y) \in X$ such that $f(x,y) = (x,y)$. (a detailed argument without explicit (i.e. non graphical) construction will give half points).

Exercise 4 Recall Sperner Lemma and its proof.

Exercise 5 Recall the proof of : "If a correspondence Φ from C to C, a convex and compact subset of $\mathbf{R}^n$ has open pre-images and is irreflexive (i.e. $x \notin co\Phi(x)$ for every $x \in C$) then Φ admits a maximal element (i.e. there exists $x \in C$ such that $\Phi(x) = \emptyset$.)"

Exercise 6 Give the proof of Brouwer theorem which uses topological degree.