Answer the questions in the EXAM sheets.

Name (Family name, First name): ________________
Program: ________________

1. Let X be a finite set of outcomes and let $P = \left\{p : X \to [0,1] \middle| \sum_{x\in X} p(x) = 1\right\}$. Let $\succsim$ be a binary relation on set P.
   (a) (1 point) Suppose that $\succsim$ satisfies vNM axioms, weak order, continuity and independence. Specify the three vNM axioms on P.
   (b) (1 point) We say a function $U : P \to \mathbb{R}$ an *expected utility function* if there exists a vNM utility u on X such that $U(p) = \sum_{x\in X} p(x)u(x)$ for all $p \in P$. Show that expected utility function satisfies independence axiom.
   (c) (2 points) Show that, if $v : X \to \mathbb{R}$ is defined as $v(x) = au(x) + b$ where $a > 0$ and $b \in \mathbb{R}$, $U(p) = \sum_{x\in X} p(x)u(x)$ and $V(p) = \sum_{x\in X} p(x)v(x)$ represent the same preference $\succsim$. (Recall that U represents $\succsim$ if $p \succsim q \Leftrightarrow U(p) \geq U(q)$.)

2. Consider Allais Paradox

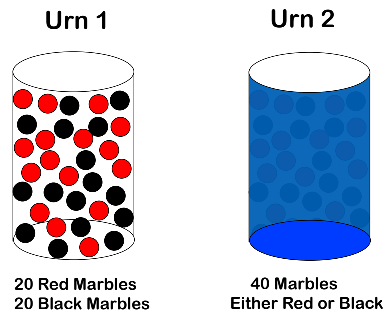
Table 1: Allais Paradox

(0.11, €1M; 0.89, €0)	$\prec$	(0.1,€5M; 0.9 €0)
(1, €1M)	$\succ$	(0.10,€5M; 0.89, €1M; 0.01,€0)

   (a) (1 point) Prove that above preference violates independence axiom.
   (b) (2 points) Consider original Prospect theory: $U(p) = \sum_{x\in X} w(p(x))u(x)$ where $w : [0,1] \to [0,1]$ is an increasing probability distortion function with $w(0) = 0$ and $w(1) = 1$. Specify a function w and demonstrate that your version of prospect theory can explain Allais Paradox.
   (c) (1 point) What is the critical problem of original prospect theory?
   (d) (2 points) To solve the problem of original prospect theory, we introduce *rank-dependent* expected utility. Specify the formal expression of rank-dependent expected utility for general lottery $p \in P$, where P is defined as in Problem 1.

3. Construct the Ellsberg example in a formulation of uncertainty. As in the graph, both Urn 1 and Urn 2 contains 40 marbles, which are either *red* or *black*. Urn 1 contains 20 red and 20 black marbles. But, we do not know the number of red (black) marbles in the Urn 2. Let us take a ball out of each Urn. Consider 4 possible act. Act 1, written f_1, is a bet that if the ball out of Urn 1 is red, then you will get 100€; if the ball out of Urn 1 is black, then you will get 0€. Act 2, written f_2, is a bet that if the ball out of Urn 1 is black, then you will get 100€; if the ball out of Urn 1 is red, then you will

get 0€. Act 3, written f_3, is a bet that if the ball out of Urn 2 is red, then you will get 100€; if the ball out of Urn 2 is black, then you will get 0€. Act 4, written f_4, is a bet that if the ball out of Urn 2 is black, then you will get 100€; if the ball out of Urn 2 is red, then you will get 0€.



(a) (1 point) Formally specify acts f_1, f_2, f_3, f_4. (To do so, you need to first specify the *states of nature*. Then the *set of outcome*. Finally, an act is a mapping from states to outcomes.)

(b) (1 point) Suppose a preference $\succsim$ satisfies $f_1 \sim f_2 \succ f_3 \sim f_4$. Demonstrate that this preference cannot be consistent with a probability belief. (You can assume this preference has an expected utility function and, then, derive a contradiction.)

(c) (2 points) To solve the Ellsberg paradox, we introduce *maxmin expected utility* theory: $U(f) = \min_{p\in\mathcal{P}} \int_S u(f(s))\mathrm{d}p(s)$, where u is vNM utility on outcomes X and $\mathcal{P}$ is a convex and closed set of priors (probabilities) on states S. Specify a set of priors $\mathcal{P}$ and vNM utility u. Show that your specified maxmin expected utility can explain the Ellsberg paradox.

(d) (2 points) To characterize maxmin expected utility, we need to weakened the independence axiom. Specify the weakened independence axiom that we introduced in class.

4. (4 points) In market, there is one risk-free asset which is money with constant price 1, and one ambiguous asset with iid normal distribution $\tilde{v} \sim N(\bar{v}, \sigma)$,

$$\bar{v} \in [v_{\min}, v_{\max}] \quad \text{and} \quad \sigma \in [\sigma_{\min}, \sigma_{\max}].$$

Investors follow maxmin expected utility rule, in which they have CARA utility for wealth $u(w) = -\exp(-w)$. Suppose initial wealth is w. Then, budget constraint is $w = m + px$, in which m is quantity of money, p is price of asset, and x is quantity of asset. So, next period wealth is $\tilde{w} = m + \tilde{v}x$. What is the optimal demand function of ambiguous asset, $x^*(p)$?