Master MMMEF, 2020-2021
Final Exam on:
General Equilibrium Theory:
Economic analysis of financial markets
December 2020

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Q1) What is the definition of an Arrow Security?
Q2) What is the definition of a redundant asset?
Q3) Under the assumption that $p(\xi) \neq 0$ for all $\xi \in \mathbb{D}$, provide a necessary and sufficient condition under which $V(p)$ is complete.
Q4) Let (p, q) be a spot - asset price pair such that q is arbitrage free. How can we choose a price $\pi \in \mathbb{R}^{\mathbb{L}}$ such that the financial budget set $B^{\mathcal{F}}(p, q)$ is included in the Walrasian budget set $B^W(\pi, \pi \cdot e_i)$?
Q5) What is the definition of the over hedging price of an asset for a given financial structure?

Exercise 1 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$, the set of states of nature at date 1, is equal to $\{\xi_1, \xi_2, \xi_3, \xi_4\}$. The financial structure is composed of two nominal assets with the following payoff matrix:

$$V = \begin{pmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}$$

1) Represent graphically in $\mathbb{R}^2$ the set Z^+ of portfolios z such that $Vz \geq 0$.
2) Show that q is an arbitrage free portfolio if and only if $q \cdot z > 0$ for all $z \in Z^+ \setminus \{0\}$.
3) Represent graphically the set of arbitrage free portfolios.

Exercise 2 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1$ is the set of states of nature at date 1. We assume that we have a unique commodity at each state. We consider a financial structure $\mathcal{F}$ with a nonempty finite collection of $\mathcal{J}$ assets, represented by the payoff mapping

$p \to V(p)$ from $\mathbb{R}^{\mathbb{D}}$ to the set of $\sharp\mathbb{D}_1 \times \mathcal{J}$ matrices. We assume that $p(\xi) > 0$ for all $\xi \in \mathbb{D}$.

Let k be an asset whose payoffs are $(v_k(p,\xi))_{\xi\in\mathbb{D}_1}$. We consider the financial structure $\tilde{\mathcal{F}}$ obtained by adding this new asset to the structure $\mathcal{F}$: the collection of assets of $\tilde{\mathcal{F}}$ is $\mathcal{J} \cup \{k\}$ and the $\sharp\mathcal{J}$ first columns of the payoff matrix $\tilde{V}(p)$ are the columns of the matrix $V(p)$ and the last column is the column of the payoffs of the asset k:

$$\tilde{V}(p) = \Big(V(p) \ \ \vdots \ \ (v_k(p,\xi))_{\xi\in\mathbb{D}_1} \Big)$$

1) Show that if $\tilde{q} = ((q_j)_{j\in\mathcal{J}}, q_k)$ is arbitrage free for the structure $\tilde{\mathcal{F}}$ at p, then the asset price $(q_j)_{j\in\mathcal{J}}$ is arbitrage free for the structure $\mathcal{F}$ at p.

2) Show that the financial structures $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p if and only if the payoff vector $(v_k(p,\xi))_{\xi\in\mathbb{D}_1}$ belongs to the range of $V(p)$.

3) Show that if q is an arbitrage free asset price for the structure V at p and the structures V and $\tilde{V}$ are equivalent at p, then there exists a unique asset price q_k for the asset k such that $\tilde{q} = (q, q_k)$ is arbitrage free for the structure $\tilde{V}$ at p.

4) Show that if the financial structure $\mathcal{F}$ is complete at p, then $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p.

We assume now that there is no redundant asset for the financial structure $\mathcal{F}$ at the price p.

5) Show that the financial structures $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equivalent at p if and only if the financial structure $\tilde{\mathcal{F}}$ has a useless portfolio.

Exercise 3 We consider a two-period model with the uncertainty represented by the graph $\mathbb{D}$. $\mathbb{D}_1 = \{\xi_1, \ldots, \xi_K\}$ is the set of states of nature at date 1. We assume that we have a unique commodity at each state. We consider a financial structure $\mathcal{F}$ with a nonempty finite collection $\mathcal{J} = \{1, \ldots, J\}$ of nominal assets defined as follows. Asset 1 has a positive payoff $v_k > 0$ for all $\xi_k \in \mathbb{D}_1$. Then there exists $0 < k_1 < k_2 < \ldots < k_{J-1} < K$ and the payoffs of Asset j at node ξ_k is 0 if $k \le k_{j-1}$ and v_{ξ_k} otherwise. So the payoff matrix is as follows:

$$V = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_1} & 0 & 0 & \dots & 0 \\ v_{(k_1+1)} & v_{(k_1+1)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_2} & v_{k_2} & 0 & \dots & 0 \\ v_{(k_2+1)} & v_{(k_2+1)} & v_{(k_2+1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_3} & v_{k_3} & v_{k_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{(k_{J-1}+1)} & v_{(k_{J-1}+1)} & v_{(k_{J-1}+1)} & \dots & v_{(k_{J-1}+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_K & v_K & v_K & \dots & v_K \end{pmatrix}$$

1) Show that the payoff matrix V of this financial structure is one-to-one.

2) Show that the set of no arbitrage asset prices is

$$Q = \{q \in \mathbb{R}^{\mathcal{J}}_{++} \mid q_1 > q_2 > \ldots > q_J\}$$

Hint: you can start by showing that $Q \subset \{q \in \mathbb{R}^{\mathcal{J}}_{++} \mid q_1 > q_2 > \ldots > q_J\}$ and then show the converse inclusion.

3) Show that the financial structure $\mathcal{F}$ is complete if and only if $K = J$ and $k_1 = 1$, $k_2 = 2$, ..., $k_{J-1} = J - 1$.

4) Show that the financial structure $\mathcal{F}$ is equivalent to the financial structure $\tilde{\mathcal{F}}$ associated to the following payoff matrix:

$$\tilde{V} = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{k_1} & 0 & 0 & \dots & 0 \\ 0 & v_{(k_1+1)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & v_{k_2} & 0 & \dots & 0 \\ 0 & 0 & v_{(k_2+1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & v_{k_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v_{(k_{J-1}+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v_K \end{pmatrix}$$