Master's Thesis

Optimal Trade Execution under Market Impact

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Contents

List of Tables

List of Figures

Abstract

In the thesis a novel approach to optimal execution problem is considered. It is based on incorporating the quadratic penalization of the remaining and terminal inventory in the objective criterion. Under the assumption of additive market impact with prices following the Arithmetic Brownian Motion, the resulting problem is demonstrated to fit into the classical Linear-Quadratic optimal control framework. It allows to derive analytical solutions for the optimal strategies. Additionally, the method of solving numerically the resulting optimization problems is described. It is used to obtain and test the results for the case of multiplicative market impact with prices driven by Geometric Brownian Motion.

1   Introduction

Along with the progressive computerization in finance industry and the development of advanced algorithmic trading methods the problem of optimal execution of large orders has become crucial for institutional investors. Since (large) transactions influence the price of an underlying asset, usually in an adverse manner, such trades incur additional costs linked to market impact. Despite its complex structure, in most cases this effect can be optimized by splitting the order into smaller transactions and executing them over a longer period. This strategy on the other hand exposes the investor to market risk connected with price fluctuations caused by other factors. Therefore the problem of optimal execution is non-trivial and its mathematical formulation is a key factor affecting the solution and the resulting strategy.

Academic research regarding this problem started after the work of [Bertsimas and Lo, 1998] who understood the best strategy as the one minimizing the expected trading costs. Their approach was later extended in [Almgren and Chriss, 1999] and [Almgren and Chriss, 2000] by including the price risk in the optimization criterion. This was a significant improvement which designated a direction for future researchers. Based on those cornerstones the optimal execution literature evolved in many different directions. Within a few years this topic, or in general, the topics linked to liquidity, gained significant popularity and they started to require a certain formalization. This direction was appointed first in [Huberman and Stanzl, 2004] where authors investigate the possibilities of exploitation of market impact leading to a weak form of arbitrage. Another significant branch of optimal execution literature is the expected utility framework which can be seen as an alternative to standard expected-revenue based problems. This issue is treated for example in [Schied and Schöneborn, 2009] or [Schied et al., 2010]. Market impact framework is used in a variety of different applications from hedging to pricing large blocks of shares as in [Guéant, 2015]. Obviously there is a large part of literature which can be considered a direct descendant of classical articles from the turn of the century. In principle it concerns extensions of standard models using different criteria, slightly modified model assumptions etc. This last group is the most closely related to the content of this thesis.

The following Section 2 contains a brief overview of the fundamentals of optimal execution literature as well as a description of some of the key concepts. In Section 3 we describe the main tools of dynamic optimization which are later used for solving the resulting problems. Subsequently the market impact model is established using an alternative form of penalization on the remaining and terminal inventory. This form of penalization intuitively enforces the strategies to satisfy certain regularity conditions. Assuming an additive form of market impact and Arithmetic Brownian Motion price dynamics we manage to explicitly solve the model and provide closed form expressions for inventory and trading rate that constitute the optimal strategy. Similarly we formulate the problem for multiplicative market impact framework with prices driven by Geometric Brownian Motion. We use this framework in order to present a numerical method which can be applied to solve the resulting problems. We then investigate the result of the numerical analysis and provide conclusions.

2   Overview of the problem formulation

After almost three decades of research the issue of optimal trade execution seems to have a certain established framework in the literature. In this section some fundamentals and basic notions are described.

Let's consider an investor who seeks to change his position from the initial amount Q_0 to some target Q_T, within a finite time horizon $[0, T]$. It is important to notice that for such a problem only the absolute size of the resulting trade matters and hence without loss of generality Q_T can be set as equal to zero. This additionally implies that from the perspective of mathematical modelling it is irrelevant if the investor wants to acquire or sell assets. In other words the situation is completely symmetrical for buying and selling with Q_0 either negative or positive, respectively. That is why it is a common practice to consider only a problem of complete liquidation of a position Q_0, which is done later in the paper as well.

The setup of the market impact model comes down to the choice of

- time framework (discrete or continuous)

- unaffected price dynamics

- form of market impact

- optimization criterion

The trade execution strategy in a market impact model can be described by an amount of assets owned at each time $t \in [0, T]$, called the inventory. It is denoted as a path $(q_t)_{t \in [0,T]}$. In many applications it is assumed that $q_T = Q_T$ i.e. the strategy always leads to complete execution. There are two main reasons behind this assumption; Firstly, it can significantly simplify the model and its solution. Secondly, it is consistent with the statement of the problem, or - in other words - the initial goal of an investor is considered to be binding. This can be the principle for most institutional investors who use execution algorithms for managing their already planned transactions. It is nevertheless tempting to add this degree of generality to the model and allow q_T to be any quantity. We can simply imagine a situation in which during an ongoing execution a trader decides to change his original aim in the view of evolving market conditions. According to this reasoning, the optimal execution strategy could be seen not as an implementational device but rather as a method designating the optimal behaviour with the desire to e.g. liquidate a position. In that case, apart from the optimal execution problem, an additional asset management problem emerges. However this poses a question how to enter the terminal inventory into the optimization criterion. It is important to notice that if the remaining position did not appear in the optimization problem it would always be optimal to execute completely under any circumstances. Thus with reasonable modelling assumptions the potential benefit of keeping some terminal inventory should be additionally penalized in order to incentivize complete execution. As a result, strategies which do not achieve the pre-established target would only be optimal in some specific scenarios. Similar, yet a bit different approach can be found for example in [Forsyth, 2011] where the authors effectuate the last trade of remaining position q_{T^-} which effectively forces the optimal strategy to satisfy $q_T = 0$.

2.1 Discrete vs continuous time

First papers dealing with optimal executions considered a problem formulated in discrete time i.e. where trading is only possible at some specified time points. This approach was developed with an idea to mimic the real market which also functions in discrete time manner. Additionally this framework has some advantages in terms of numerical simulations and solutions. Thus from a practical point of view it is useful to work with models in discrete time and due to their functionality they are still present in the modern literature concerning optimal execution. On the other hand, from purely mathematical perspective it is more convenient to work with models in continuous time. Continuous models are not necessarily far from reality since with the development of high-frequency trading, transactions can occur in very short time intervals, as if they took place in continuous time. Furthermore, modern optimal executions algorithms used by institutional investors and traders consist of at least two layers. Following the nomenclature from [Guéant, 2016] they can be seen as a strategic and tactical one. The former acts as a scheduling layer which establishes a general path to follow while executing an order. The latter deals with micromanagement of orders. This means that its main goal is to arrange orders among different trading platforms to optimize transaction costs, find the best possible price available etc. This thesis among most of other papers concerning the topic of optimal execution examines the problem faced by the strategic layer i.e. to schedule an optimal trading curve with some assumptions

about the current market conditions. From this point of view it seems to be more convenient to work with continuous time models which can be later discretized if there is such a need. For that reason the model in the following sections is presented in continuous time. Contrariwise, in most of the early literature on the topic of optimal execution a discrete time framework is used. It is motivated by the fact that formerly the prevailing objective was to build an execution algorithm as a whole, without distinguishing the strategic layer from the tactical one. Consequently the principal question was how to perform a trade, rather than how to only schedule it.

2.2 Price dynamics

The underlying price process is another crucial component of the market impact model. First of all it may be worth to understand that in real financial market due to its complexity, the variety on trading venues etc. there is no single price of a given asset. The usual understanding of the price is the mid price (mid point between highest bid and lowest ask offers), but as a matter of fact it can be any price, e.g. the average of bid and ask orders, currently available buy/sell price, or other. The definition of the price can be taken into consideration in the model formulation (for example in [Almgren and Chriss, 2000] or [Forsyth et al., 2012] authors add a supplementary trading cost connected to the bid-ask spread). However in general it is not essential to specify what exactly the assumed price process represents and presumably it may play a more significant role from the practical point of view.

In this thesis the price of the underlying asset at time t is denoted by P_t . It represents the unaffected price, meaning that this is the assumed price dynamics without any transactions made by an investor. In mathematical finance the prices of risky assets are usually assumed to follow a Geometric Brownian Motion (GBM) dynamics, that is to satisfy the following Stochastic Differential Equation (SDE):

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \tag{2.1}$$

The solution of this SDE is the following

$$P_t = P_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{2.2}$$

where W_t is a standard Brownian Motion (BM).

The above dynamics means that prices are log-normally distributed, or in other words, that price returns (relative changes in price) are normally distributed.
However in the optimal execution literature an Arithmetic Brownian Motion (ABM) is frequently used for modelling the price of the underlying asset. ABM is defined by the SDE:

$$dP_t = \mu dt + \sigma dW_t, \tag{2.3}$$

which implies that

$$P_t = P_0 + \mu t + \sigma W_t, \tag{2.4}$$

Under ABM dynamics prices are normally distributed and have independent, identically distributed increments. The reason for which the ABM can be preferred is that the model is usually much simpler to work with, even though an ABM can be regarded as a simplistic, unrealistic dynamic of the asset price, especially in highly volatile conditions. Moreover, it also admits non-zero probabilities of negative prices which can be seen as a certain drawback. On the other hand, optimal order executions are by definition short-term problems and hence, over a considered time span, price movements are in general not notably significant. This argument can be regarded as a response for aforementioned issues because over the trading horizon:

1. the probability of the price becoming negative is extremely small

2. in the case of reasonably low volatility ABM is a plausible approximation of GBM dynamics

The comparison between ABM and GBM in the view of optimal execution has also been discussed in [Forsyth et al., 2012] where authors conduct a variety of numerical experiments and point out that, surprisingly, the optimal strategies under ABM prices are very similar to those obtained with assumed GBM dynamics, even in the case of high volatility. Although this result could be closely related to the specific formulation of the model studied in that paper, it still provides a suggestion that ABM dynamics can be a solid assumption in optimal execution problems. Nevertheless, in order to remedy the conceptual downsides of ABM framework researchers made a number of attempts to obtain or generalize solutions to fit the GBM framework. For example in [Gatheral and Schied, 2011] an alternative risk measure is incorporated in the model which allows to derive a closed-form expression for optimal trade execution strategy. In [Baldacci and Beneveniste, 2020] the authors slightly modify the form of market impact and manage to explicitly solve the resulting problem under GBM assumption even in the case of stochastic drift.

In general, including a drift term in the price dynamics can considerably hinder the calculations. That is why it is quite common in the optimal execution literature to disregard it i.e. take $\mu = 0$. The rationale behind this assumption can be once again linked to the short-time horizon of the problem, because it is reasonable to suppose that the trader does not possess any information regarding the price movements in near future. This implies that the unaffected price is a martingale, which means that $\mathbb{E}[P_t|P_s] = P_s$ for $t > s$). However a possibility to include a drift term in price diffusion can be a handy feature and whenever it is doable many studies provide extensions for that case as well.

2.3 Market impact

The crux of optimal execution topic is a question how to model the market impact. The nature of how trading can affect prices has been studied for a long time before first optimal execution problems were formulated as it is a crucial factor for asset managers, brokers and investors. In reality the exact mechanics of market impact can be very complex and distinct in different market conditions. Moreover, it is difficult to extract even the size of impact for a given transaction due to stochastic fluctuations of the price. In other words it is not known explicitly whether the price changed because of the transaction itself, or it was triggered by other factors. Some of these issues, as well as the techniques which allow to estimate the size of market impact, are described for example in [Almgren et al., 2005].

For the reasons mentioned above, it is impossible in practice to find functional forms which could precisely correspond to real-life scenarios of market impact. Therefore it seems reasonable to choose relatively easy functions which could make the problem tractable and at the same time would satisfy some stylized facts about the market impact.

The approach proposed in [Almgren and Chriss, 2000] fulfilled both conditions and over the years it has become a standard in the literature. The authors based their model on the assumption that price impact can be decomposed into permanent and temporary component. In other words they assumed that each transaction triggers two effects: one which is persistent on the market and thus it affects equally all current and future trades, and the second one which affects only the current trade and disappears instantaneously. This idea has been suggested also in some previous studies regarding market impact and additionally, it seemed to be at least partly confirmed by some empirical results. However the main accomplishment of Almgren and Chriss was that they proposed a clear formalization of this simple yet very general idea and incorporated it into their framework of optimal execution. This led to broad acceptance of their approach by academics and professionals, and even though some more complicated frameworks have been developed, the division of market impact into two components is somehow present in most papers. Traditional interpretation of this fragmentation also originates from [Almgren and Chriss, 2000]. The authors argued that the permanent impact corresponds to a reveal of information in the market while temporary component stands for short-term depletion of orders in limit order book. In simple words, even if large transactions could be executed in a very short period of time (the asset is liquid enough) there is a limited number of orders at the "market price". However, modern understanding of the temporary component is slightly different and it is linked to the aforementioned distinction between strategic and tactical layers of execution algorithms. As explained in [Guéant, 2016], the initial idea of Almgren and Chriss was to construct an entire execution strategy using market orders. Nowadays their approach is used rather to design a scheduling layer only and

so temporary impact represents the average cost of trading which cannot be accounted to permanent impact. This definition is broader, including many other factors such as transactions costs, bid-ask spread, fees, taxes etc. That is why in the literature the instantaneous component of market impact is often referred to as execution costs. It is worth to highlight that these costs are estimated taking into account the possibility to maneuver among several venues or trading platforms, using different types of orders etc, so in fact they represent the *average* cost of trading. In particular this indicates that even though the scheduling layer of an algorithm includes the expected execution costs, the tactical layer is the one to optimize them.

Once the above consensus is established there is still a question how to incorporate market impact functions into price dynamics. There are two dominant approaches in the literature. In the first one market impact influences the price of the underlying asset in an additive manner. This approach was formalized in [Almgren and Chriss, 2000] and for that reason in the literature it is usually referred to as Almgren & Chriss model (of market impact). It has been vastly adopted both by the academic community and the financial industry. Its popularization was caused by the fact that this framework enabled to develop first closed-form solutions to optimal liquidation problems which included both price risk and optimization of execution costs. In [Almgren and Chriss, 2000] the authors also provided an explicit formula for the optimal strategy for their formulation of the problem. Their result is a non-trivial strategy which can be computed before the beginning of execution process. This serves exactly the purpose of scheduling a trading curve to follow during the execution. The main drawback of an additive framework is the fact that very fast executions could drive the price to become negative. In order to address this unrealistic property of the additive model the authors of [Bertsimas and Lo, 1998] introduced the multiplicative version of the model as an alternative. They also pointed out that in the additive model the percentage price impact, that is the market impact expressed as a percentage of execution price, is decreasing in price level, which is counterfactual as suggested for example in [Loeb, 1983]. The multiplicative model resolves that issue as well. The disadvantage of the multiplicative framework is the difficulty to find explicit solutions. Nevertheless it is still widely considered in the literature.

Last but not least, it is necessary to choose the functional form of both the permanent and temporary market impact components. A standard assumption for the permanent impact is a linear function of the trading rate. There exist strong theoretical indications behind it. Namely it has been proven that with linear permanent impact function the model does not admit price manipulation strategies. The proof is due to [Gatheral, 2010] and [Forsyth et al., 2012] for additive and multiplicative framework, respectively. In simple words, price manipulation can be described as a strategy such that $q_T = q_0 = 0$ (a round trip) with positive expected revenues. Intuitively for some forms of market impact the investor can benefit from the predicted price fluctuations triggered by his own trades through market impact and manipulating the execution velocities in time can lead on average to gaining profits. Hence the absence of price manipulation strategies is a highly desirable property of the market impact model. Moreover, according to some empirical estimations choosing a linear function can be considered as reasonably realistic for the purpose of modelling (this is suggested in [Almgren et al., 2005] for an additive model). Needless to say, this functional form also makes the model more tractable and in many cases it makes the closed-form solutions available. For the latter reason the temporary price impact function is in many studies assumed to be linear too. However the general consensus is to set it as a power function. In case of temporary impact there are no theoretical indications as to which functional form should be chosen and hence power functions seem like a balanced choice in terms of flexibility and simplicity. In some papers authors consider also additional components to temporary impact corresponding to bid/ask spread (which in principle does not depend on the trading rate). It has been done for example in [Forsyth, 2011].

2.4 Optimization problem

The optimal execution problem can be stated either as some form of maximization of revenues obtained from the strategy or as minimization of the expected trading costs. The revenues of a sell program can be simply defined as a sum of period-to-period incomes earned by selling portions of an asset at the current execution price S_t^X. Consequently, in continuous time framework, by heuristic argument the revenues would be a sum over infinitesimal amounts of asset $-dq_t$ multiplied by the price

S_t^X. Thus total revenues within the time horizon $[0, T]$ can be expressed as $-\int_0^T S_t^X dq_t$. The mapping $t \longrightarrow -\int_0^t S_s^X dq_s$ can be referred to as the cash process. Additionally the dynamics of portfolio value or cash process can be altered by a risk-free rate if some form of compounding is included in the model. However due to the short-time horizon argument the optimal execution problem can be treated as if the risk-free rate was equal to zero, which is done in most papers. Subsequently, the total liquidation cost is the quantity $S_0 q_0 + \int_0^T S_t^X dq_t$, where S_0 is the price of the asset before any trading occurs and thus $S_0 q_0$ stands for the par value of the initial portfolio (without accounting for market impact effects). Since $S_0 q_0$ is a constant it does not change the optimization problem and clearly the minimization of expected costs is completely equivalent to maximizing expected revenues.

This basic criterion i.e. minimization of expected cost was considered initially in [Bertsimas and Lo, 1998]. As a result the authors obtained a so called naive strategy, meaning that a portfolio is liquidated at a constant rate during the whole trading period. In other words the optimal strategy is simply $q_t = \frac{t}{T} q_0$ and the *trading rate*, which is the velocity of selling, is $\dot{q}_t = \frac{q_0}{T}$. This optimization problem is the core of many more complex criteria however by itself it does not take into account the price risk. This issue has been addressed in two alternative ways:

1. adding some form of risk penalty

2. using expected utility

2.5 Risk criteria

A classical risk penalty introduced in [Almgren and Chriss, 2000] is a penalization of the variance of total revenues (equivalently total costs). The mean-variance criterion is a standard in finance as it is used in many problems, for example in Markowitz portfolio theory (see [Markowitz, 1952]). The advantages of this criterion are definitely the clarity and intuitiveness. However later research on optimal execution exposed some subtleties which have to be taken into consideration. First of all the issue with mean-variance criterion is that the solution derived in [Almgren and Chriss, 2000] is strongly dependent on the assumed ABM dynamics of unaffected price. In order to come up with a solution in different frameworks of the problem e.g. with GBM dynamics, some other criteria were proposed. An example could be [Gatheral and Schied, 2011] where authors propose another risk penalty, namely time averaged value at risk, which they argue to be similarly intuitive and practical as variance. The second problem with mean-variance formulation is that variance seen at the initial time is not a time consistent risk measure which have some implications for truly optimal strategies especially when the dynamic programming methods are used for problem solving. For that reason in some papers authors choose to use quadratic variation instead of variance.

2.6 Deterministic vs adaptive strategies

A very intriguing question is whether the investor should optimize his behaviour over the deterministic strategies or dynamic ones. Consequently one more distinction should be made in order to be precise in that matter. As it is highlighted in [Almgren and Lorenz, 2007] in fact there are three different types of strategies:

1. deterministic or static strategy, which meaning that the optimal trading curve can be computed before the trading period and does not adapt during the execution process

2. *pre-commitment* strategy, meaning that it is described by a certain optimal rule determined before trading and this rule stays unchanged during the whole execution. This implies in principle that the strategy can depend on some state variables unknown at initial time (e.g. the realisation of price process)

3. dynamic strategies which are adaptive during the execution process, meaning that the risk criterion is recalculated at each time during execution and strategy is adjusted

Intuitively the possibility to adapt the strategy is desired and could lead to higher revenues. However as it is discussed in the following section it strongly depends on the optimization problem and the definition of *optimality* which strategy would be the best. Apart from that, in some cases it can be

desirable to consider the set of strategies which are static or path-independent. In particular when an investor needs to determine a trading curve before the execution process begins.

2.7 Time-consistency

The difference in definition of pre-commitment and dynamic/adaptive strategy is crucial when it comes to time-inconsistent optimization criteria. In order to solve for dynamical strategy the standard approach is to use the tools of dynamic optimization and primarily apply the Bellman principle of optimality which states that at any point in time the remaining policy is optimal with respect to the state at that time. This means that a dynamic policy obtained in such a way is time-consistent. In case of time-inconsistent optimization problem, developing an adaptive strategy with dynamic programming techniques leads to strategies different from pre-commitment ones. Those dynamic strategies are then suboptimal in terms of the optimization criterion as seen at the initial time. Following the argumentation from [Forsyth, 2011], in a idealized world in which all modelling assumptions are met the pre-commitment strategy provides on average the best results in terms of the optimization criterion as seen at the start of execution. It is worth to mention that it is exactly what an institutional investor expects from an execution algorithm. Moreover, with time-inconsistent criterion the pre-commitment strategy leads to time-inconsistent policy which can be beneficial. With time-consistent criteria this is not the problem since pre-commitment strategy is the same as the dynamic one.

3 Dynamic programming

This section contains a concise description of dynamic programming method used for solving various optimization problems and in particular (stochastic) optimal control problems. The results gathered is this section are presented in a simplified way, only to the extent required to formulate the basic framework which will be used in next sections of the thesis. The content of this chapter is based mainly on [Pham, 2009] and [Fleming and Rishel, 1975] and the reader is referred to those books for a more profound study of this topic.

Let's establish a filtered probabilistic space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ and consider a controlled system whose dynamic is characterized by a SDE of the form

$$\begin{cases} dX_s = b(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s, & (t \leq s \leq T) \\ X_t = x \end{cases} \tag{3.1}$$

where: W is the standard d-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, X is the process valued in $\mathbb{R}^n$ describing the evolution of state variables and $\alpha = (\alpha_s)_s$ is a progressively measurable process, valued in $A \subset \mathbb{R}^m$ which represents the control (we denote $\alpha \in \mathbb{H}(A, (F_t)_t)$). Moreover let's restrict our attention to the set of *admissible* controls which additionally satisfy the condition

$$\mathbb{E}\left[\int_t^T |\alpha_s| ds\right] \leq \infty$$

so that $\int_t^T |\alpha_s| ds$ belongs to a set of almost-surely bounded random variables $L^\infty(\Omega)$. Moreover functions $b : [0, T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n$ are continuous and measurable. In order to ensure the existence of a unique, strong solution to the SDE (3.1) $\forall \alpha \in \mathcal{A}$ and for any initial state $(t, x) \in [0, T] \times \mathbb{R}^n$, it is imposed that b and σ satisfy the uniform Lipschitz condition, i.e. $\forall (s, y), (t, x) \in [0, T] \times \mathbb{R}^n$ such that $s \geq t$, $\forall a \in A$

$$|b(s, y, a) - b(t, x, a)| + |\sigma(s, y, a) - \sigma(t, x, a)| \leq C(|s - t| + |y - x|)$$

for some suitable constant C.

The optimization problem is to find a control $(\alpha_s)_s$ maximizing the payoff

$$\mathbb{E}\left[\int_0^T r(s, X_s, \alpha_s)ds + g(T, X_T)\right] \tag{3.2}$$

for some measurable functions $r : [0,T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}$ and $g : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the following quadratic growth conditions

$$\begin{cases} r(s,x,a) \leq C(1 + |x|^2 + |a|^2), & \forall (x,a) \in \mathbb{R}^n \times A \\ g(s,x) = C(1 + |x|^2), & \forall x \in \mathbb{R}^n \end{cases} \tag{3.3}$$

where C is some suitable constant.

Function r and g are referred to as a running payoff and a terminal payoff, respectively.

Objective functional and the value function

In order to solve (3.2) let's consider a broader class of problems given any initial state $(t,x) \in [0,T] \times \mathbb{R}^n$ and define an *objective functional*

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T r(s, X_s, \alpha_s) ds + g(T, X_T) \right] \tag{3.4}$$

.

The objective in the broader sense is to maximize the objective functional over the set of admissible controls. Let's define the *value function* as

$$V(t,x) = \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha) \tag{3.5}$$

.

3.1 Principle of optimality

The (Bellman's) principle of optimality was first introduced in [Bellman, 1957] but it is also often referred to as dynamic programming principle because it provides a foundation to determining the decision policy when the initial problem is split into smaller subproblems. According to this principle the optimal policy consists of the optimal decisions in each of the subproblems. Formally we write that for any initial state $(t,x) \in [0,T] \times \mathbb{R}^n$

$$V(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^{t+h} r(s, X_s, \alpha_s) ds + V(t+h, X_{t+h}) \right] \tag{3.6}$$

where X_{t+h} is understood as a state resulting from the dynamics (3.1) under the optimal control $(\alpha_s)_{s \in [t, t+h]}$.

3.2 Dynamic programming Partial Differential Equation

On the basis of the Bellman's principle of optimality one can derive a Partial Differential Equation (PDE) satisfied by the value function. It is particularly useful because finding the solution to this PDE, along with a suitable verification theorem (see for example [Pham, 2009], provides also a solution to the initial optimization problem (3.2). Moreover it can also be used to construct an optimal decision policy in a form of a *feedback control*.

Let's consider an arbitrary control α, a straightforward application of (3.6) indicates that

$$V(t,x) \geq \mathbb{E}\left[\int_t^{t+h} r(s, X_s, \alpha_s) ds + V(t+h, X_{t+h}) \right] \tag{3.7}$$

Next, let's assume that a value function V is smooth enough in order to apply the Itô's lemma between t and $t+h$ so that

$$V(t+h, X_{t+h}) = V(t,x) + \int_t^{t+h} \left(\frac{\partial V}{\partial t}(s, X_s) + \mathcal{L}V(s, X_s) \right) ds + \int_t^{t+h} \sigma(s, X_s, \alpha_s) dW_s \tag{3.8}$$

where $\mathcal{L}^\alpha$ the derivative operator associated with (3.1) for the control and arbitrary control α, i.e.

$$\mathcal{L}^\alpha V = b(s, X_s, \alpha_s)\nabla_x V + \frac{1}{2}Tr(\sigma(s, X_s, \alpha_s)^T \sigma(s, X_s, \alpha_s)\nabla_x^2 V) \tag{3.9}$$

Substituting (3.8) into (3.7) gives

$$0 \geq \mathbb{E}\left[\int_t^{t+h}\left(r(s, X_s, \alpha_s) + \frac{\partial V}{\partial t}(s, X_s) + \mathcal{L}^\alpha V(s, X_s)\right)\right] \tag{3.10}$$

Dividing both sides of (3.10) by h and sending h to 0 yields

$$0 \geq \frac{\partial V}{\partial t}(t, x) + \mathcal{L}^a V(t, x) + r(t, x, a) \tag{3.11}$$

where $a = \alpha_t \in A$. Hence $\forall (t, x) \in [0, T] \times \mathbb{R}^n$ we obtain an equality in the case of the optimal α_t^* and

$$0 = \frac{\partial V}{\partial t}(t, x) + \sup_{a \in A}\{\mathcal{L}^a V(t, x) + r(t, x, a)\} \tag{3.12}$$

The PDE (3.12) is called the Hamilton-Jacobi-Bellman (HJB) equation.

4 Optimal execution with alternative penalty

In this section a market impact model is constructed. It is based on a novel problem formulation with optimization criterion taking into account a trajectory of the execution strategy. In some cases analytical solutions of the posed problem are derived.

4.1 Model specification

Let's establish a filtered probabilistic space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ and consider times t in some fixed interval $[0, T]$. The strategy is represented by an absolutely continuous, almost-surely (a.s.) bounded, progressively measurable process $q_t = q_0 - \int_0^t \dot{v}_t dt$ where $v_t = -\dot{q}_t$ is the trading rate controlled by an investor. The dynamics of the inventory is thus given by

$$dq_t = -v_t dt$$

In order to work in stochastic optimal control framework let's characterize the strategy by its trading rate $(v_t)_{t \in [0, T]} \in A$ where A is the set of admissible controls i.e.

$$\mathcal{A} := \left\{(v_t)_{t \in [0, T]} \in \mathbb{H}(\mathbb{R}, (F_t)_t), \int_0^T |v_t| dt \in L^\infty(\Omega)\right\}$$

Denote by X_t the wealth processes defined as the face value of the portfolio at time t plus the risk-free bank account B_t $(B_0 = 0)$

$$X_t = q_t S_t + B_t \tag{4.1}$$

where S_t is the price of the asset. In particular this means that S_t is affected by the permanent market impact (i.e. by all trades up to time t).
Cash obtained from trading is stored in the bank account hence the dynamics of B are given by

$$dB_t = (rB_t + v_t S_t^X)dt \tag{4.2}$$

where r is the risk-free rate. Throughout the rest of the thesis it is assumed that in the short-term

horizon the risk-free rate can be effectively chosen to be equal to 0. Furthermore S_t^X denotes the *execution price* which is the price S_t affected additionally by the temporary market impact component. Intuitively the price S_t^X is only in relation to the inventory actually traded while S_t is the *market price* of the asset. This implies that

$$X_t = q_t S_t + \int_0^t S_s^X v_s ds \tag{4.3}$$

which is simply the face value of the portfolio plus the revenues obtained from trading. It is important to highlight that the value of the portfolio is only theoretical, since due to the market impact its liquidation would bring a revenue strictly less than $q_t S_t$.

The optimization problem is to maximize the terminal wealth with additional quadratic penalties on the remaining and terminal inventory. It is assumed that an investor's goal is to execute his position, although his admissible strategies do not necessarily have to satisfy this goal. In other words, there is no limitation of the form $\int_0^T v_t dt = q_0$. The optimization problem can be written as

$$\sup_{(v_t)_t \in A} \mathbb{E} \left[X_T - \rho q_T^2 - \phi \int_0^T q_t^2 dt \right] \tag{4.4}$$

Let's observe that the introduced penalization is expected to work in two ways. Firstly, it incentivises complete execution but does not enforce it entirely. Secondly, the penalty on the terminal inventory dictates that the trading is not excessive. In other words, it sanctions strategies which do not progress directly towards the specified target i.e. liquidation. In effect, procrastination of execution, intermediate purchases or round-trip strategies are not preferred. Intuitively this form of penalization can guarantee that the chosen strategies behave properly. Additionally, it can be seen as an imposed risk-avoidance criterion since faster execution leads to reduced price risk. On the other hand, the disadvantage of that problem formulation is that risk aversion resulting from the introduced penalties cannot be directly quantified. In general it can be presumed that increasing the parameters ρ and (especially) ϕ would lead to less risky strategies, but that may be difficult to relate to standard risk measures like for example variance. For this reason it can be argued that this form of penalization by itself lacks the interpretation and hence it is not intuitive. Nevertheless the strategies corresponding to different values of the model parameters can be easily compared between each other in terms of riskiness.

4.2 Additive market impact

In this subsection an additive framework of market impact is considered, i.e.

$$\begin{aligned} S_t &= P_t + \int_0^t g(\dot{q}_s) ds \\ S_t^X &= S_t + h(\dot{q}_t) \end{aligned} \tag{4.5}$$

where P_t is the unaffected price of the asset at time t and $g(\cdot)$, $h(\cdot)$ are permanent and temporary impact functions, respectively. In the additive framework it is in general assumed throughout the thesis that the unaffected price follows an ABM dynamics.

Assuming linear impact functions i.e.

$$\begin{cases} g(\dot{q}_t) = \gamma \dot{q}_t = -\gamma v_t \\ h(\dot{q}_t) = \eta \dot{q}_t = -\eta v_t \end{cases} \tag{4.6}$$

equations (4.5) simplify to

$$\begin{aligned} S_t &= P_t + \gamma(q_t - q_0) \\ S_t^X &= P_t + \gamma(q_t - q_0) - \eta v_t \end{aligned} \tag{4.7}$$

Rewriting the terminal wealth

$$X_T = q_T S_T + \int_0^T S_t^X v_t dt$$

$$= q_T(P_T + \gamma(q_T - q_0)) + \int_0^T P_t v_t dt + \gamma \int_0^T q_t v_t dt - \gamma q_0 \int_0^T v_t dt - \eta \int_0^T v_t^2 dt$$

$$= q_T P_T + \gamma(q_T^2 - q_T q_0) - \int_0^T P_t dq_t - \gamma \int_0^T q_t dq_t + \gamma q_0 \int_0^T dq_t - \eta \int_0^T v_t^2 dt \qquad (4.8)$$

$$= q_T P_T + \gamma(q_T^2 - q_T q_0) - \int_0^T P_t dq_t - \gamma(\frac{1}{2}q_T^2 - \frac{1}{2}q_0^2) + \gamma q_0(q_T - q_0) - \eta \int_0^T v_t^2 dt$$

$$= q_T P_T - \int_0^T P_t dq_t - \eta \int_0^T v_t^2 dt + \frac{\gamma}{2}(q_T^2 - q_0^2)$$

Using Ito's integration by parts formula

$$d(P_t q_t) = P_t dq_t + q_t dP_t \qquad (4.9)$$

it can be obtained that

$$X_T = q_T P_T - \int_0^T P_t dq_t - \eta \int_0^T v_t^2 dt + \frac{\gamma}{2}(q_T^2 - q_0^2)$$

$$= q_T P_T - \int_0^T d(P_t q_t) + \int_0^T q_t dP_t + \eta \int_0^T v_t^2 dt + \frac{\gamma}{2}(q_T^2 - q_0^2)$$

$$= q_T P_T - P_T q_T + P_0 q_0 + \int_0^T q_t dP_t - \eta \int_0^T v_t^2 dt + \frac{\gamma}{2}(q_T^2 - q_0^2) \qquad (4.10)$$

$$= q_0 P_0 + \int_0^T q_t dP_t - \eta \int_0^T v_t^2 dt + \frac{\gamma}{2}(q_T^2 - q_0^2)$$

After substituting into (4.4) the problem writes as

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[q_0 P_0 + \int_0^T q_t dP_t - \eta \int_0^T v_t^2 dt + \frac{\gamma}{2}(q_T^2 - q_0^2) - \rho q_T^2 - \phi \int_0^T q_t^2 dt \right]$$

Since q_0 and P_0 are fixed constants they can be disregarded in the optimization problem, hence the problem simplifies to

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T q_t dP_t + \int_0^T (-\eta v_t^2 - \phi q_t^2) dt + (\frac{\gamma}{2} - \rho)q_T^2 \right] \qquad (4.11)$$

4.2.1   No-drift price dynamics

If there is no drift (i.e. $\mu = 0$) then (4.11) simplifies to:

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T (-\eta v_t^2 - \phi q_t^2) dt + (\frac{\gamma}{2} - \rho)q_T^2 \right] \qquad (4.12)$$

This is a standard Linear Quadratic (LQ) optimal control problem and hence finding and optimal control v_t boils down to solving the Riccati differential equation which can be determined via dynamic programming approach. The first step is to write an optimization functional for any state $(q, t) = (q_t, t)$ where $0 < t < T$

$$J(q, t, (v_s)_{s \geq t}) = \mathbb{E}\left[\int_t^T (-\eta v_s^2 - \phi q_s^2) ds + (\frac{\gamma}{2} - \rho)q_T^2 \right] \qquad (4.13)$$

15

Define a corresponding value function

$$V(q,t) = \sup_{(v_s)_{s \geq t} \in A} J(q,t,(v_s)_s) \tag{4.14}$$

The HJB equation resulting from Bellman's principle of optimality can be written as

$$0 = V_t - \phi q^2 + \sup_{v \in \mathbb{R}} \left\{ -v V_q - \eta v^2 \right\} \tag{4.15}$$

where V_t and V_q denote the derivatives of value function with respect to time and inventory q.

The terminal condition is

$$V(q,T) = (\frac{\gamma}{2} - \rho)q^2 \tag{4.16}$$

Let's observe that inside the sup there is a quadratic function of v with negative coefficient $-\eta$ guaranteeing that the maximum is in fact attained. The desired value v^* sets the derivative of the term inside the sup to be equal to zero. Hence

$$v^* = -\frac{V_q}{2\eta} \tag{4.17}$$

Inserting into (4.15) yields

$$0 = V_t - \phi q^2 + \frac{1}{4\eta}V_q^2 \tag{4.18}$$

In order to solve the PDE above let's make a following ansatz

$$V(q,t) = f(t)q^2 \tag{4.19}$$

then

$$\begin{aligned} V_t &= f'(t)q^2 \\ V_q &= 2f(t)q \\ V(q,T) &= f(T)q^2 \implies f(T) = (\tfrac{\gamma}{2} - \rho) \end{aligned} \tag{4.20}$$

Substituting into (4.15) and (5.9) yields

$$\begin{cases} 0 = f'(t)q^2 - \phi q^2 + \frac{f(t)^2}{\eta}q^2 \\ f(T) = (\frac{\gamma}{2} - \rho) \end{cases} \tag{4.21}$$

which gives the following Riccati equation

$$\begin{cases} 0 = f'(t) - \phi + \frac{f(t)^2}{\eta} \\ f(T) = (\frac{\gamma}{2} - \rho) \end{cases} \tag{4.22}$$

Once the above non-linear, first order Ordinary Differential Equation (ODE) is solved the optimal control can be obtained with the formula (4.17)

$$v^* = -\frac{V_q}{2\eta} = -\frac{f(t)}{\eta}q \tag{4.23}$$

The function f satisfying (4.22) is the following (solution in the Appendix A.1)

$$f(t) = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\phi\eta} \qquad (4.24)$$

where $M = \dfrac{(\frac{\gamma}{2}-\rho)-\sqrt{\frac{\phi}{\eta}}}{(\frac{\gamma}{2}-\rho)+\sqrt{\frac{\phi}{\eta}}}$

Then

$$v_t^* = \coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\frac{\phi}{\eta}}q_t^* \qquad (4.25)$$

which gives a first-order ODE for q_t

$$\dot{q}_t^* = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\frac{\phi}{\eta}}q_t^* \qquad (4.26)$$

A solution to (4.26) is (see Appendix A.2)

$$q_t^* = q_0 \frac{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|\right)} \qquad (4.27)$$

and the optimal trading rate expressed as a function of initial inventory is given by

$$v_t^* = q_0\sqrt{\frac{\phi}{\eta}}\frac{\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|\right)} \qquad (4.28)$$

Note that under the same assumptions the same strategy is also optimal in the under GBM unaffected price diffusion with no drift.

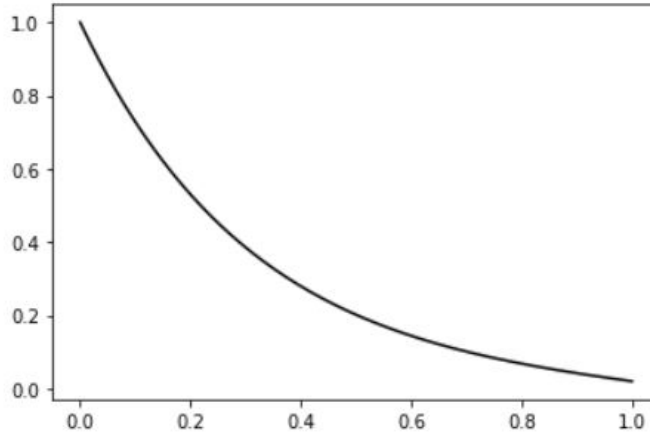An example of the trading curve defining optimal inventory q_t is presented in Figure 1.

Figure 1: Example of the optimal trading curve. (Parameters: $\gamma = 0.1$, $\rho = \phi = T = q_0 = 1$, $\eta = 0.1$)

4.2.2 Price dynamics with drift term

If there is a drift and unaffected price follows an ABM i.e. P_t is defined by (2.3). Then the problem (4.11) writes as

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T q_t\mu dt + \int_0^T q_t\sigma dW_t - \eta\int_0^T v_t^2 dt - \phi\int_0^T q_t^2 dt + (\frac{\gamma}{2} - \rho)q_T^2\right] \qquad (4.29)$$

which simplifies to:

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T (-\eta v_t^2 - \phi q_t^2 + \mu q_t)dt + (\frac{\gamma}{2} - \rho)q_T^2\right] \qquad (4.30)$$

This problem can be reformulated in order to obtain a generalized, matrix formulation of the LQ problem. Let's define the vector of state variables x, and controls α as

$$x_t = \begin{bmatrix} q_t \\ \mu \end{bmatrix} \quad \dot{x}_t = \begin{bmatrix} \dot{q}_t \\ \dot{\mu} \end{bmatrix} \quad \alpha_t = \begin{bmatrix} v_t \\ u_t \end{bmatrix} \qquad (4.31)$$

ans so the dynamics of the system are given by

$$\begin{cases} \dot{x}_s = N\alpha_s & (t \leq s \leq T) \\ x_t = [q_t, \mu]^T \end{cases} \qquad (4.32)$$

where

$$N = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.33)$$

Introducing an additional control u_t is done just for technical reasons. It is worth to notice that with matrix N defined as above it does not affect the dynamics of the system. The optimization problem can be written as

$$\sup_{(\alpha_t)_t} \mathbb{E}\left[-\int_0^T (x_t^T B x_t + \alpha_t^T C \alpha_t)dt - x_T^T D x_T\right] \qquad (4.34)$$

where

$$B = \begin{bmatrix} \phi & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \quad C = \begin{bmatrix} \eta & 0 \\ 0 & d \end{bmatrix} \quad D = \begin{bmatrix} (\rho - \frac{\gamma}{2}) & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.35)$$

so that the problem writes as

$$\sup_{(\alpha_t)_t} \mathbb{E}\left[\int_0^T (-\eta v_t^2 - \phi q_t^2 + \mu q_t - d^2 u_t)dt + (\frac{\gamma}{2} - \rho)q_T^2\right] \qquad (4.36)$$

Let's notice that an extra term $d^2 u_t$ is present in the optimization problem. For technical reasons, the matrix C is required to be invertible which implies that $d \neq 0$. Hence the solution to the problem (4.36) is equivalent to the solution of (4.30) only if when $\forall t \in [0, T] u_t = 0$. It will be later demonstrated that in fact it is the case.

Similarly as before let's define a optimization functional for any state $(x, t) = (x_t, t)$

$$J(x, t, (\alpha_t)_t) = \mathbb{E}\left[-\int_t^T (x_s^T B x_s + \alpha_s^T C \alpha_s)ds - x_T^T D x_T\right] \qquad (4.37)$$

and derive the HJB equation

$$0 = V_t - x^T B x + \sup_{a \in \mathbb{R}^2} \{ (\nabla_x V)^T N a - a^T C a \} \tag{4.38}$$

with the terminal condition

$$V(x, T) = -x^T D x \tag{4.39}$$

Let's observe that a maximizing the term inside the supremum is equal to

$$a = \frac{1}{2} C^{-1} N^T \nabla_x V \tag{4.40}$$

which after inserting into HJB (4.38) gives

$$\begin{cases} 0 = V_t + \frac{1}{4}(\nabla_x V)^T N C^{-1} N^T \nabla_x V - x^T B x \\ V(x, T) = -x^T D x \end{cases} \tag{4.41}$$

Let's make an educated guess that the value function V is of the form $V(x, t) = x^T K(t) x$ where $K(\cdot)$ is some proper, symmetric, differentiable matrix. This implies that

$$V_t = x^T \dot{K}(t) x$$
$$\nabla_x V = 2K(t)x \tag{4.42}$$
$$V(q, T) = x^T K(T) x \implies K(T) = -D$$

Inserting into (4.38) yields

$$0 = x^T [\dot{K}(t) + K(t) N C^{-1} N^T K(t) - B] x \tag{4.43}$$

which gives the following matrix Riccati equation

$$\begin{cases} 0 = \dot{K}(t) + K(t) N C^{-1} N^T K(t) - B \\ K(T) = -D \end{cases} \tag{4.44}$$

and optimal control $(\alpha_t^*)_t$ is given by

$$\alpha_t^* = C^{-1} N^T K(t) x_t \tag{4.45}$$

Let's write the matrix K as

$$K(t) = \begin{bmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{bmatrix} \tag{4.46}$$

because of the fact that K is symmetric one can substitute $k_{21} = k_{12}$. Furthermore inserting for matrices N, C and B into the matrix Riccati equation (4.44) gives the following system of ODEs

$$\begin{cases} \dot{k}_{11}(t) = -\frac{1}{\eta} k_{11}^2(t) + \phi \\ k_{11}(T) = (\frac{\gamma}{2} - \rho) \\ \dot{k}_{12}(t) = -\frac{1}{\eta} k_{11}(t) k_{12}(t) - \frac{1}{2} \\ k_{12}(T) = 0 \\ \dot{k}_{22}(t) = -\frac{1}{\eta} k_{12}^2(t) \\ k_{22}(T) = 0 \end{cases} \tag{4.47}$$

Moreover the optimal control is given by

$$\alpha_t^* = \begin{bmatrix} -\frac{1}{\eta}k_{11}(t)q_t - \frac{\mu}{\eta}k_{12}(t) \\ 0 \end{bmatrix}$$ (4.48)

It can be observed that k_{11} coincides with the function f computed in 4.2.1, that is

$$k_{11}(t) = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\phi\eta}$$ (4.49)

where $M = \frac{(\frac{\gamma}{2}-\rho)-\sqrt{\frac{\phi}{\eta}}}{(\frac{\gamma}{2}-\rho)+\sqrt{\frac{\phi}{\eta}}}$

Having the functional form of $k_{11}(t)$ the second ODE can be solved in order to get the formula for $k_{12}(t)$. The solution writes as (see Appendix A.3)

$$k_{12}(t) = \frac{\sqrt{\eta}}{2\sqrt{\phi}}\left(\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right) - \frac{\cosh\left(\frac{1}{2}\ln|M|\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}\right)$$ (4.50)

and hence the formula for optimal trading rate $(v_t^*)_t$ is given by

$$v_t^* = \coth(\kappa_t)\sqrt{\frac{\phi}{\eta}}q_t^* - \frac{\mu}{2\sqrt{\eta\phi}}\left(\coth(\kappa_t) - \frac{\cosh\left(\frac{1}{2}\ln|M|\right)}{\sinh(\kappa_t)}\right)$$ (4.51)

where $\kappa_t = \sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|$

The optimal inventory path can be obtained by solving the ODE (4.51) and the solution writes as (see Appendix A.4)

$$q_t = \left(q_0 - \frac{\mu}{2\phi}\right)\frac{\sinh(\kappa_t)}{\sinh(\beta)} + \frac{\mu}{2\phi} + \frac{\mu}{2\phi}\cosh\left(\frac{1}{2}\ln|M|\right)\left(\frac{\sinh\left(-\sqrt{\frac{\phi}{\eta}}t\right)}{\sinh(\beta)}\right)$$ (4.52)

and optimal trading rate as a function of initial inventory can be expressed as

$$v_t = \left(q_0 - \frac{\mu}{2\phi}\right)\frac{\cosh(\kappa_t)}{\sinh(\beta)}\sqrt{\frac{\phi}{\eta}} + \frac{\mu}{2\sqrt{\phi\eta}}\cosh\left(\frac{1}{2}\ln|M|\right)\left(\frac{\cosh(\kappa_t)\sinh\left(-\sqrt{\frac{\phi}{\eta}}t\right) + \sinh\beta}{\sinh(\kappa_t)\sinh(\beta)}\right)$$ (4.53)

where $\kappa_t = \sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|$, $\beta = \sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|$

It has to be highlighted that in order to formally prove that the control defined by (4.53) solely is optimal for the problem (4.30) one should compute the value function using the equation $V(x,t) = x^T K(t)x$ and then use verification type argument.

Figure 2 contains the comparison of the exemplary trading curves defining optimal inventory q_t resulting from cases with and without the assumption about the drift term in the price process.
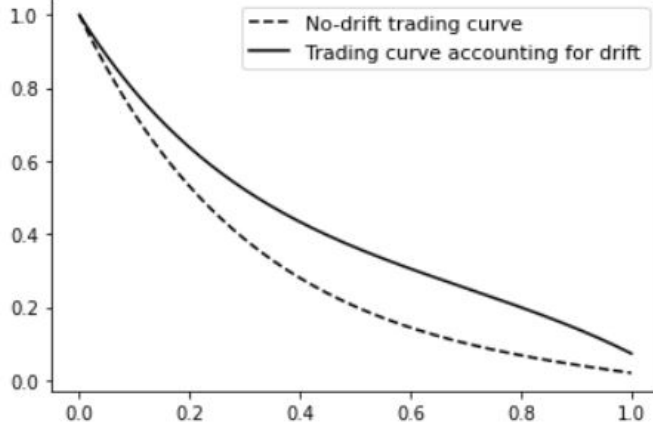
Figure 2: Comparison of the optimal trading curves with or without accounting for the drift term. (Parameters: $\gamma = 0.1$, $\rho = \phi = T = q_0 = 1$, $\eta = 0.1$, $\mu = 0.5$)

4.3 Multiplicative market impact

In multiplicative framework the price of an asset is given by:

$$
\begin{aligned}
S_t &= P_t e^{\int_0^t g(\dot{q}_s)ds} \\
S_t^X &= S_t e^{h(\dot{q}_t)}
\end{aligned}
\tag{4.54}
$$

For simplicity we denote $\pi(\cdot) = e^{h(\cdot)}$ so that execution price can be expressed as

$$
S_t^X = \pi(\dot{q}_t)S_t
\tag{4.55}
$$

Since multiplicative framework is often considered to eliminate the possibility of price becoming negative it usually comes with the assumption of unaffected price following a GBE dynamics defined by (2.1). This implies that:

$$
S_t = P_0 e^{\int_0^t (\mu - \frac{\sigma^2}{2} + g(\dot{q}_s))ds + \sigma W_t}
\tag{4.56}
$$

which together with the assumption of linear impact function (see (4.6)) gives:

$$
\begin{aligned}
S_t &= P_0 e^{\int_0^t (\mu - \frac{\sigma^2}{2} - \gamma v_s)ds + \sigma W_t} \\
S_t^X &= S_t e^{-\eta v_t} = \pi(v_t)S_t
\end{aligned}
\tag{4.57}
$$

with $\pi(v_t) = e^{-\eta v_t}$. This implies the following dynamics of price S_t:

$$
dS_t = (\mu - \gamma v_t)S_t dt + \sigma S_t dW_t
\tag{4.58}
$$

Inserting the terminal wealth into (4.4) yields the following optimization problem:

$$
\sup_{(v_t)_t \in A} \mathbb{E}\left[S_T q_T + \int_0^T S_t v_t dt - \rho q_T^2 - \phi \int_0^T q_t^2 dt \right]
$$

which can be written as:

21

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T (\pi(v_t)S_t v_t - \phi q_t^2)dt + q_T(S_T - \rho q_T)\right]$$

Let's define objective functional for some initial state $(s, q, t) = (S_t, q_t, t)$ where $0 < t < T$

$$J(S, q, t, (v_t)_t) = \mathbb{E}\left[\int_t^T (\pi(v_s)S_s v_s - \phi q_s^2)ds + q_T(S_T - \rho q_T)\right] \tag{4.59}$$

The corresponding value function is

$$\hat{V}(S, q, t) = \sup_{(\hat{v}_t)_t \in A} \mathbb{E}\left[\int_t^T (\pi(v_s)S_s v_s - \phi q_s^2)ds + q_T(S_T - \rho q_T)\right] \tag{4.60}$$

Define $V(S, q, \tau = T - t) = \hat{V}(S, q, t)$ then the optimal control $(v_t)_t$ satisfies the following HJB equation

$$V_\tau = \frac{1}{2}S^2\sigma^2 V_{SS} + \mu S V_S - \phi q^2 + \sup_{v \in \mathbb{R}}\{vS\pi(v) - vV_q - \gamma v S V_S\} \tag{4.61}$$

with the initial condition

$$V(S, q, \tau = 0) = q(S - \rho q) \tag{4.62}$$

In Section 6 it is demonstrated that this equation can be solved using numerical methods.

5 Simplified version of the problem

Let's now consider a similar approach to optimal execution but this time restricting the set of admissible strategies to satisfy additionally the condition of complete execution i.e. $\int_0^T v_t = q_0$. This simplifies the model in two ways. Firstly, the terminal wealth is now equal to cash revenues obtained by trading and secondly, the penalty for not hitting the target becomes redundant.

Let's consider the probabilistic space $(\Omega, \mathcal{F}, \mathcal{F}_{t \in [0,T]}, \mathbb{P})$ in fixed time horizon $[0, T]$.

The set of admissible strategies is

$$\mathcal{A}^* := \left\{(v_t)_{t \in [0,T]} \in \mathcal{H}(\mathbb{R}, (F_t)_t), \int_0^T v_t dt = q_0, \int_0^T |v_t| dt \in L^\infty(\Omega)\right\}$$

The optimization problem of an investor is

$$\sup_{(v_t)_t \in A^*} \mathbb{E}\left[X_T - \phi \int_0^T q_T^2 dt\right] \tag{5.1}$$

for wealth process defined by (4.3)

5.1 Additive market impact

In this subsection an additive framework of market impact defined in (4.5) is considered. It is moreover assumed that impact functions are linear and given by (4.6), so that the execution price at time t is equal to

$$S_t^X = P_t + \gamma(q_t - q_0) - \eta v_t \tag{5.2}$$

where P_t is the unaffected price of the asset at time t.

Using the fact that $q_T = 0$ the terminal wealth becomes:

$$X_T = q_0 P_0 - \frac{\gamma}{2} q_0^2 + \int_0^T q_t dP_t - \eta \int_0^T v_t^2 dt \tag{5.3}$$

and the problem (5.1) becomes

$$\sup_{(v_t)_t \in A^*} \mathbb{E}\left[\int_0^T q_t dP_t - \eta \int_0^T v_t^2 dt - \phi \int_0^T q_t^2 dt\right] \tag{5.4}$$

5.1.1 No-drift price dynamics

If there is no drift (i.e. $\mu = 0$) then (5.4) simplifies to:

$$\sup_{(v_t)_t \in A^*} \mathbb{E}\left[\int_0^T (-\eta v_t^2 - \phi q_t^2) dt\right] \tag{5.5}$$

This problem is clearly similar to the one considered in 4.2.1 and it will become evident that the solution can be obtained by following the same steps.

The optimization functional for any state $(q, t) = (q_t, t)$ where $0 < t < T$ can be written as

$$J(q, t, (v_s)_{s \geq t}) = \mathbb{E}\left[\int_t^T (-\eta v_s^2 + -\phi q_s^2) ds\right] \tag{5.6}$$

A corresponding value function can be defined as

$$V(q, t) = \sup_{(v_s)_{s \geq t} \in A} J(q, t, (v_s)_s) \tag{5.7}$$

and the HJB equation can be expressed as

$$0 = V_t - \phi q^2 + \sup_{v \in \mathbb{R}} \left\{-v V_q - \eta v^2\right\} \tag{5.8}$$

Due to the complete execution constraint the terminal condition can be written as:

$$V(q, T) = \begin{cases} -\infty & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \tag{5.9}$$

This singular terminal condition can be interpreted as an infinite penalty for the states in which the complete liquidation has not been fulfilled. As a consequence, the optimal strategy would always restrain itself from reaching those states.

Let's observe that the HJB equation is the same as (4.15), but with different terminal condition. Hence the ansatz in this case is the following

$$\begin{cases} V(q, t) = f(t) q^2 \\ V(q, T) = \psi(q) q^2 \end{cases} \tag{5.10}$$

where

$$\psi(q) = \begin{cases} -\infty & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \tag{5.11}$$

Following the same steps as in 4.2.1 gives the Riccati differential equation

$$\begin{cases} 0 = f'(t) - \phi + \frac{f(t)^2}{\eta} \\ f(T) = \psi(q) \end{cases} \tag{5.12}$$

The solution is equivalent as in Appendix A.1. The function f solving (5.12) is

$$f(t) = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\phi\eta} \tag{5.13}$$

where M now writes as

$$M = \frac{\psi(q) - \sqrt{\frac{\phi}{\eta}}}{\psi(q) + \sqrt{\frac{\phi}{\eta}}} \tag{5.14}$$

Since $\psi(q)$ is equal either to $-\infty$ or 0 necessarily $|M| = 1$ and $\ln|M| = 0$, the solution simplifies to

$$f(t) = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)\sqrt{\phi\eta} \tag{5.15}$$

Accordingly as in 4.2.1 function f can be used to solve for optimal inventory and optimal trading rate which can be expressed as

$$q_t^* = q_0 \frac{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T\right)} \tag{5.16}$$

$$v_t^* = q_0\sqrt{\frac{\phi}{\eta}}\frac{\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T\right)} \tag{5.17}$$

Let's observe that this is the same strategy as defined previously in (4.27) and (4.28) for the limit case when $\rho \longrightarrow +\infty$. This is very intuitive as it can expected that with the increase of the penalty on terminal inventory the optimal strategy would tend to the fulfill complete execution.

5.1.2 Price dynamics with drift

In the case when a drift term is assumed not to be equal to 0 and unaffected price follows an ABM the optimization problem writes as

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T q_t\mu dt + \int_0^T q_t\sigma dW_t - \eta\int_0^T v_t^2 dt - \phi\int_0^T q_t^2 dt\right] \tag{5.18}$$

which can be simplified to

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T \mu q_t dt - \int_0^T \eta v_t^2 dt - \phi\int_0^T q_t^2 dt\right] \tag{5.19}$$

and

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[-\int_0^T \eta v_t^2 dt - \phi\int_0^T (q_t^2 - \frac{\mu}{\phi}q_t)dt\right] \tag{5.20}$$

which reformulates as:

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[-\int_0^T \eta v_t^2 dt - \phi \int_0^T (q_t - \frac{\mu}{2\phi})^2 dt + \frac{\mu^2}{4\phi}T\right] \tag{5.21}$$

Let's introduce a new process which is defined as

$$y_t = q_t - \frac{\mu}{2\phi} \tag{5.22}$$

then the dynamics are given by

$$\begin{cases} dy_t = dq_t \\ y_0 = q_0 - \frac{\mu}{2\phi} \end{cases} \tag{5.23}$$

Thus after inserting for $v_t = -\dot{y}_t$ and excluding the constants the optimization problem writes as

$$\sup_{(\dot{y}_t)_t \in A} \mathbb{E}\left[\int_0^T (-\eta \dot{y}_t^2 - \phi y_t^2) dt\right] \tag{5.24}$$

Note that this problem is equivalent to the one considered previously in 5.1.1. Hence the solution is given by

$$y_t^* = y_0 \frac{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T\right)} \tag{5.25}$$

$$\dot{y}_t^* = -y_0 \sqrt{\frac{\phi}{\eta}} \frac{\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T\right)} \tag{5.26}$$

and substituting $y_t^* = q_t^* - \frac{\mu}{2\phi}$ and $v_t^* = -\dot{y}_t^*$ gives explicit formulas for optimal trading rate and inventory path

$$q_t^* = \left(q_0 - \frac{\mu}{2\phi}\right) \frac{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T\right)} + \frac{\mu}{2\phi} \tag{5.27}$$

$$v_t^* = \left(q_0 - \frac{\mu}{2\phi}\right) \sqrt{\frac{\phi}{\eta}} \frac{\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T\right)} \tag{5.28}$$

It is important to notice that using this method subtly changes the considered problem which after substitution becomes the portfolio transition rather than portfolio liquidation. In other words the target terminal inventory in the optimal strategy is not 0 but $\frac{\mu}{2\phi}$. In order to obtain a true complete execution strategy one can consider a limiting case of the solution determined in 4.2.2 and hence it is of the form

$$q_t^{true} = \left(q_0 - \frac{\mu}{2\phi}\right) \frac{\sinh(\kappa_t)}{\sinh(\beta)} + \frac{\mu}{2\phi} + \frac{\mu}{2\phi}\left(\frac{\sinh\left(-\sqrt{\frac{\phi}{\eta}}t\right)}{\sinh(\beta)}\right)$$

$$v_t^{true} = \left(q_0 - \frac{\mu}{2\phi}\right) \frac{\cosh(\kappa_t)}{\sinh(\beta)} \sqrt{\frac{\phi}{\eta}} + \frac{\mu}{2\sqrt{\phi\eta}}\left(\frac{\cosh(\kappa_t)\sinh\left(-\sqrt{\frac{\phi}{\eta}}t\right) + \sinh(\beta)}{\sinh(\kappa_t)\sinh(\beta)}\right)$$

where $\kappa_t = \sqrt{\frac{\phi}{\eta}}(T-t)$, $\beta = \sqrt{\frac{\phi}{\eta}}T$.

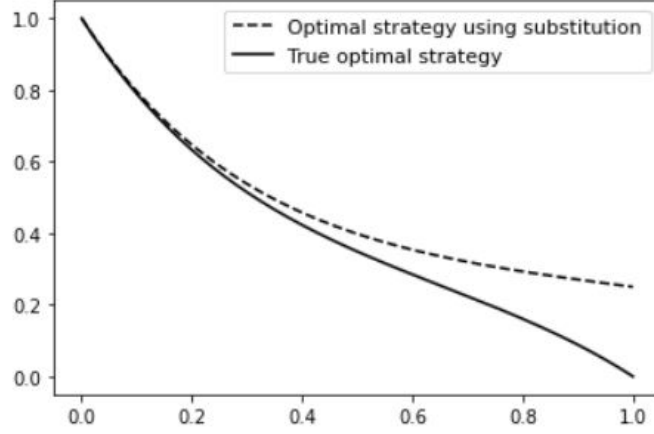This issue is presented in Figure 3.

25

Figure 3: Comparison of the trading curves obtained with substitution and by considering the limiting case of general solution. (Parameters: $\gamma = 0.1$, $\rho = \phi = T = q_0 = 1$, $\eta = 0.1$, $\mu = 0.5$)

5.2 Multiplicative market impact

Once again let's consider a multiplicative market impact framework with linear impact function defined in (4.6). The unaffected price is still assumed to follow a GBM dynamics, i.e. (2.1), and hence the equations defining the market and execution price are the same as (4.57). The dynamics of S_t are given by

$$dS_t = (\mu - \gamma v_t)S_t dt + \sigma S_t dW_t \tag{5.29}$$

In the simplified, constrained version of the problem the terminal wealth rewrites as

$$X_T = \int_0^T S_t v_t dt = \int_0^T \pi(v_t)Z_t v_t dt \tag{5.30}$$

And problem (5.1) becomes

$$\sup_{(v_t)_t \in A} \mathbb{E}\left[\int_0^T (\pi(v_t)Z_t v_t - \phi q_t^2)dt\right] \tag{5.31}$$

The objective functional for some initial state $(S, q, t) = (S_t, q_t, t)$ where $0 < t < T$ is

$$J(S, q, t, (v_t)_t) = \mathbb{E}\left[\int_t^T (\pi(v_s)S_s v_s - \phi q_s^2)ds\right] \tag{5.32}$$

The corresponding value function is

$$\hat{V}(S, q, t) = \sup_{(v_t)_t \in A} \mathbb{E}\left[\int_t^T (\pi(v_s)S_s v_s - \phi q_s^2)ds\right] \tag{5.33}$$

Define $V(S, q, \tau = T - t) = \hat{V}(S, q, t)$ then the optimal control $(v_t)_t$ satisfies the following HJB

$$V_\tau = \frac{1}{2}S^2\sigma^2 V_{SS} + \mu S V_S - \phi q^2 + \sup_{v \in \mathbb{R}}\{vS\pi(v) - vV_q - \gamma v S V_S\} \tag{5.34}$$

with initial condition

$$V(S, q, \tau = 0) = \begin{cases} -\infty & \text{if } q \neq 0 \\ 0 & \text{if } q = 0 \end{cases} \tag{5.35}$$

6 Numerical experiments

The aim of this section is to present a method which can be used to solve numerically problems introduced in previous chapters. In particular it can be applied to obtain solutions in multiplicative market impact framework presented in 4.3 and 5.2. Determining optimal strategies comes to solving the resulting HJB equations (6), and extracting the optimal trading rate for some randomly generated price process. Let's notice that the HJB equations in the unconstrained, original problem and in the simplified case are the same up to the terminal condition. That is, we consider an equation

$$V_\tau = \frac{1}{2}S^2\sigma^2 V_{SS} + \mu S V_S - \phi q^2 + \sup_{v \in \mathbb{R}}\left\{vSe^{-\eta v} - vV_q - \gamma v S V_S\right\} \tag{6.1}$$

with the initial condition

$$V(S, q, \tau = 0) = \begin{cases} q(S - \rho q) & \text{in the unconstrained problem} \\ 0 & \text{in the simplified case} \end{cases} \tag{6.2}$$

In order to solve this equation an adaptation of the method presented in [Forsyth, 2011] is used. It is based on utilizing the semi-Lagrangian scheme which means that the problem is reformulated using the Lagrangian derivative, describing the rate of change of the value function along a trajectory defined by the evolution of price and inventory determined by the optimal control (trading rate). This trajectory can be summarized with the following equations

$$\begin{aligned} \frac{dq}{d\tau} &= v \\ \frac{dS}{d\tau} &= (\gamma v - \mu)S \end{aligned} \tag{6.3}$$

The Lagrangian derivative corresponding to the trajectory (6.8) is expressed as

$$\frac{DV}{D\tau}(v) = V_\tau + (\gamma v - \mu)SV_S + vV_q \tag{6.4}$$

Hence the equation (6.1) can be reformulated as

$$\sup_{v \in \mathbb{R}}\left\{-\frac{DV}{D\tau}(v) + vSe^{-\eta v}\right\} + \frac{1}{2}S^2\sigma^2 V_{SS} - \phi q^2 = 0 \tag{6.5}$$

6.1 Discretization

The equation (6.5) can be solved using the discretization approach. In order to do that let's localize the domain of the value function $[0, +\infty] \times [q_{min}, q_{max}] \times [0, T] \ni (S, q, T)$ to be able to work in the narrowed space which can be discretized. Let's denote this localized domain by

$$\Theta = [0, s_{max}] \times [q_{min}, q_{max}] \times [0, T] \tag{6.6}$$

for some specified set of parameters $s_{max}, q_{min}, q_{max}$ (and T). Similarly let's restrict the interval of admissible controls to $[v_{min}, v_{max}]$. For the purpose of the following computations consider equally spaced subdivisions

$$[s_0, s_1, \ldots, s_i, \ldots, s_{i_{max}}] \quad \text{with} \quad s_i = i\Delta s, \quad (s_0 = 0, s_{i_{max}} = s_{max})$$

$$[q_0, q_1, \ldots, q_j, \ldots, q_{j_{max}}] \quad \text{with} \quad q_{j+1} = q_j + \Delta q, \quad (q_0 = q_{min}, q_{j_{max}} = q_{max})$$

$$[\tau_0, \tau_1, \ldots, \tau_n, \ldots, \tau_{n_{max}}] \quad \text{with} \quad \tau_n = n\Delta\tau, \quad (\tau_0 = 0, \tau_{n_{max}} = T)$$

so that the space Θ can be approximated as a regular three-dimensional grid. For each point on that grid let's consider an approximation $V_{i,j}^n$ of the true value $V(s_i, q_j, \tau_n)$ and write $v_{i,j}^n$ for the

approximation of the control at that point. The partial derivatives are approximated using standard finite difference method. The Lagrange derivative at a point (s_i, q_j, τ_{n+1}) can be approximated by

$$\left(\frac{DV}{D\tau}(v_{i,j}^{n+1})\right)_{i,j}^{n+1} \approx \frac{1}{\Delta\tau}\left(V_{i,j}^{n+1} - V_{s^n,q^n}^n(v_{i,j}^{n+1})\right) \tag{6.7}$$

where $V_{s^n,q^n}^n(v_{i,j}^{n+1})$ is a projection backwards in time of $V_{i,j}^{n+1}$, given the control $v_{i,j}^{n+1}$. In other words V_{s^n,q^n}^n stands for the approximation of $V(s^n, q^n, \tau_n)$ where s^n, q^n are defined by

$$
\begin{aligned}
s^n &= s_i + (\mu - \gamma v_{i,j}^{n+1})s_i\Delta\tau \\
q^n &= -v_{i,j}^n q_j\Delta\tau
\end{aligned}
\tag{6.8}
$$

An intuitive explanation of this approximation procedure is that applying the control $v_{i,j}^{n+1}$ at point (s_i, q_j, τ_{n+1}) results in a transition to the point (s^n, q^n, τ_n) and hence the change of V along this trajectory is $V(s_i, q_j, \tau_{n+1}) - V(s^n, q^n, \tau_n)$. It is important to mention that the projected point (s^n, q^n, τ_n) might not coincide with any point on the defined grid and in that case V_{s^n,q^n}^n is linearly interpolated using the discrete values $V_{i,j}^n$.

Altogether this gives the discretization of the equation

$$V_{i,j}^{n+1} - \underbrace{\frac{s_i^2\sigma^2\Delta\tau}{2\Delta s^2}}_{c_i}\left(V_{i+1,j}^{n+1} - 2V_{i,j}^{n+1} + V_{i-1,j}^{n+1}\right) = \underbrace{\sup_{v_{i,j}^{n+1}\in\mathbb{R}}\left\{V_{s^n,q^n}^n(v_{i,j}^{n+1}) + \Delta\tau v_{i,j}^{n+1}s_i e^{-\eta v_{i,j}^{n+1}}\right\}}_{d_{i,j}^{n+1}} - \Delta\tau\phi q_j^2$$

Values $d_{i,j}^{n+1}$ are computed using the following procedure.
Let's define

$$h = \frac{\Delta s}{C_1} = \frac{\Delta q}{C_2} = \frac{\Delta\tau}{C_1} \tag{6.9}$$

for some positive constants C_m. We discretize the interval $[v_{min}, v_{max}$ using a spacing h to obtain a finite sequence of candidates for optimal control $v_{i,j}^{n+1}$. Then for each point $(s_i, q_j, \tau^{n+1}$ a linear search through this discretized interval is applied in order to find the trading rate that maximizes the expression inside the sumpremum. According to [Wang and Forsyth, 2008], this method of solving the local optimization problem guarantees convergence to the viscosity solution of the HJB equation (for the formal definition as well as a study on the existence and uniqueness of viscosity the reader is referred to [Fleming and Soner, 1993]). Furthermore, for each point (s_i, q_j, τ^{n+1}) the control space is additionally restricted so that the resulting inventory stays within the bounds $[q_{min}, q_{max}]$. It is implemented by setting a large, negative value of V_{s^n,q^n}^n once the projected inventory q^n exceeds the boundaries. This method guarantees that corresponding trading rates are never optimal. Moreover it is assumed that at s_{max} the effects of permanent market impact can be neglected and so s^n is replaced by s_{max} once $s^n \geq s_{max}$. These two assumptions enforce that selected optimal trading rate $v_{i,j}^{n+1}$ does not entail transition into regions outside of the considered space Θ.

Let's make use of the fact that $c_0 = 0$ and rewrite the discretized equation into a matrix form

$$B\mathbf{V}_j^{n+1} - \mathbf{p}V_{i_{max},j}^{n+1} = \mathbf{d}_j^{n+1} - Q_j \tag{6.10}$$

where

$$\mathbf{V}_j^{n+1} = \begin{bmatrix} V_{0,j}^{n+1} \\ \vdots \\ V_{i_{max-1},j}^{n+1} \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} 0 \\ \vdots \\ c_{i_{max}} \end{bmatrix} \quad \mathbf{d}_j^{n+1} = \begin{bmatrix} d_{0,j}^{n+1} \\ \vdots \\ d_{i_{max-1},j}^{n+1} \end{bmatrix} \quad Q_j = \begin{bmatrix} \Delta\tau\phi q_j^2 \\ \vdots \\ \Delta\tau\phi q_j^2 \end{bmatrix} \tag{6.11}$$

and where B is a matrix with $[-c_1, \ldots, -c_{i_{max-1}}]$ on the subdiagonal, $[1 + 2c_0, \ldots, 1 + 2c_{i_{max-1}}]$ on the diagonal and $[-c_0, \ldots, -c_{i_{max-2}}]$ on the upperdiagonal, i.e.

$$B = \begin{bmatrix} 1 + 2c_0 & -c_0 & \ldots & & 0 \\ -c_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & -c_{i_{max-2}} \\ 0 & \ldots & -c_{i_{max-1}} & 1 + 2c_{i_{max-1}} \end{bmatrix}$$

Moreover B is a diagonally dominant matrix which guarantees its non-singularity and hence equation (6.10) can be rewritten as

$$\mathbf{V}_j^{n+1} = B^{-1} \left[\mathbf{d}_j^{n+1} - Q_j + \mathbf{p} V_{i_{max},j}^{n+1} \right] \tag{6.12}$$

For each j the expression (6.12) can be used to determine the approximated values of V on the defined discretized grid by iterating over time. In order to do that it is necessary to specify the boundary condition for values at s_{max} as well as use the initial condition at $\tau = 0$.

6.2 Boundary conditions

At $\tau = 0$ the boundary condition is defined by (6.2).

At s_{max} it is assumed that V is linear in S and hence $V_S S = 0$ and the $S V_S = V$, this implies that the HJB (6.1) simplifies to

$$V_\tau = \mu S V - \phi q^2 + \sup_{v \in \mathbb{R}} \left\{ v S e^{-\eta v} - v V_q - \gamma v V \right\} \tag{6.13}$$

which can be discretized using explicit Euler scheme as

$$\frac{V_{i_{max},j}^{n+1} - V_{i_{max},j}^n}{\Delta \tau} = \mu V_{i_{max},j}^n - \phi q_j^2 + \sup_{v \in \mathbb{R}} \left\{ v s_{i_{max}} e^{-\eta v} - \gamma v V_{i_{max},j}^n - v \frac{V_{i_{max},j+1}^n - V_{i_{max},j}^n}{\Delta q} \right\} \tag{6.14}$$

and so

$$V_{i_{max},j}^{n+1} = (\Delta \tau \mu + 1) V_{i_{max},j}^n - \Delta \tau \phi q_j^2 + \Delta \tau \sup_{v \in \mathbb{R}} \left\{ v s_{i_{max}} e^{-\eta v} - \gamma v V_{i_{max},j}^n - v \frac{V_{i_{max},j+1}^n - V_{i_{max},j}^n}{\Delta q} \right\} \tag{6.15}$$

where for tractability v denotes $v_{i_{max},j+1}^{n+1}$. Moreover V_q at $q_{j_{max}}$ is approximated by the corresponding value at $q_{j_{max-1}}$. The linearity of V with respect to S at s_{max} is a simplifying assumption and thus the behaviour of V at that boundary has to be carefully analyzed while performing computations. Using the equation (6.15) it is possible to obtain explicitly $V_{i_{max},j}^{n+1}$ at each iteration, which can be used to compute values $V_{i,j}^{n+1}$ for any (i, j).

From the computational point of view it is efficient to consider equations (6.12) and (6.15) as three-dimensional with additional dimension indexed by j (inventory) rather than iterate over each j.

6.3 Results

The procedure of obtaining the results is the following. Firstly, the boundary initial condition is specified according to (6.2). Then at each iteration the value of V at the boundary s_{max} is determined (i.e. $V_{i_{max},j}^{n+1}$) and the iteration is finished with calculation of $V_{i,j}^{n+1}$ for any point on the grid using (6.12). As a byproduct the approximated values of optimal controls are obtained simultaneously to V as arguments maximizing the local optimization problem. Once the values of V and v are determined

for each point on the grid, the unaffected price process is simulated using the discretized version of equation (2.1) and given some specified initial price S_{init}.

In order to extract the optimal trading rate for the corresponding realization of unaffected price iteratively simulate the path of (q_t, S_t, t) with the initial point $(q_{init}, S_{init}, t = 0)$. At each iteration the corresponding trading rate v_t is used to the determine the point in the next period. Similarly as before if the point $(S_t + 1, q_t + 1, t + 1)$ does not coincide with any point on the grid it is linearly interpolated to obtain v^{t+1}.

With this method we generate the whole trajectory of S_t, q_t along with extracting the optimal control sequence. Moreover we can additionally compute the penalties using the inventory path and discretizing the integral $\int q_t^2 dt$.

All the results presented in this thesis were obtained in Python environment by direct application of the ensemble of the methods described above.

Let's consider a different subdivisions of the considered grid and model parametrizations summarized in Tables 1 and 2 respectively.

	Subdivision 1	Subdivision 2	Subdivision 3
s_{max}	20	40	80
i_{max}	80	160	320
q_{min}	0	0	0
q_{max}	10	20	40
j_{max}	60	120	240

Table 1: Subdivisions

	Parameterization 1	Parameterization 2	Parameterization 3	Parameterization 4
S_{init}	5	5	5	5
q_{init}	5	5	5	5
T	0.1	0.1	0.1	0.1
n_{max}	100	100	100	100
μ	0.1	0.1	0.1	0.1
σ	0.1	0.1	0.1	0.1
γ	0.02	0.02	0.02	0.02
η	0.002	0.0005	0.002	0.002
ϕ	0.5	0.5	3	0
ρ	1	1	0	1
v_{min}	-500	-500	-500	-500
v_{max}	2000	2000	2000	2000
h	10	10	10	10

Table 2: Parameterizations

h denotes the spacing of the interval $[v_{min}, v_{max}]$ and $n_{max}+1$ corresponds to the number of considered time points. It is worth to notice that even though there is a finite set of candidates for the optimal trading rate, the values v can be in fact spread across the whole interval due to the linear interpolation which is used to determine them. For that reason the spacing h is only crucial for determining the values at points which lie on the grid.

In all of the experiments the same simulated path of Brownian Motion is used (demonstrated in Figure 4). It is worth to mention that this is the only random factor driving the model.
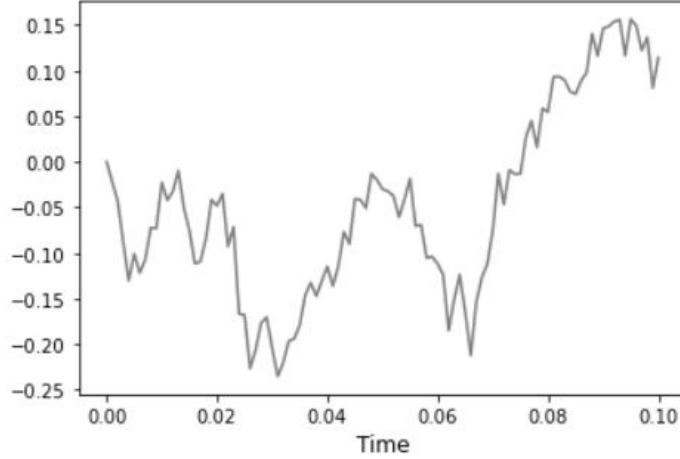
Figure 4: Realisation of Brownian Motion

Stability & convergence

Firstly, the results were studied in terms of stability with respect to the chosen bounds for the space Θ. For this test the *Parameterization 1* was used along with each of the subdivisions from table 1. The results are summarized in Tables 3 and 4 respectively for the problem with and without the complete execution constraint.

Subdivision	$V(S_{init}, q_{init}, t = 0)$	Penalty	Gain
1	1.67985e+08	0.19519	1.67985e+08
2	21.23819	0.84243	22.08062
3	21.23819	0.84243	22.08062

Table 3: Value function and penalty for *Parameterization 1* with different subdivisions under the full-execution constraint.

Subdivision	$V(S_{init}, q_{init}, t = 0)$	Penalty	Gain
1	1292380.609	0.04231	1292380.651
2	21.00799	0.39555	21.40354
3	21.00799	0.39555	21.40354

Table 4: Value function and penalty for *Parameterization 1* with different subdivisions under the full-execution constraint.

Investigating the result above it can be noticed that when the boundaries are too small the solution becomes extremely incorrect. The reason for that may be linked to the approximation at the boundary at s_{max} which is in that case close to the regions of interest, at the same time affecting significantly the results. This statement is motivated by the inspection of the behaviour of the value function at boundary s_{max} which indicated that after a few iterations the function takes unreasonably big values which continue to surge along the time axis. It can be roughly estimated that at each point $(s_i, q_j, \cdot)$ function V should not exceed the value $s_i q_j$ which represent the face value of currently held assets without accounting for market impact factors. Values at and near the boundary s_{max} do not satisfy that criterion. Moreover, analysis of the optimal trading rates obtained with too small boundaries in general suggest that they behave in a similar manner. Namely, near the initial time they tend to oscillate between extreme values v_{min} and v_{max}.

Based on the results presented in 3 and 4 it can be stated that in regions distant from the boundary the calculated values are reasonable. On the contrary, this would also imply that once the simulated path falls into the region near the boundary s_{max} the solution is false and impractical, because it entails uncontrolled behaviour of the optimal control sequence. On the other hand it is sufficient to choose

boundaries which are not too widespread and the solution is then stable in the regions of interest. For that reason the *Subdivision 2* was used as the base in other experiments unless otherwise specified.

It is worth to mention that the considered solution is price-dependent and hence we can only calculate a gain *a posteriori*, given the price realisation. This in principle means that the computed gain depends on whether the price increases or decreases and it may not be coherent with the actual expected gain.

In order to answer the question whether the incorrect results obtained for *Subdivision 1* were linked to the small boundary or rather to the discretization of the intervals $[0, s_{max}$ and $[q_{min}, q_{max}$ another experiment was conducted. Notice that in each subdivision the number of considered points was proportionally adjusted to the length of the interval so that Δs and Δq are the same in all the cases. In the second experiment the *Parameterization 1* and *Subdivision 1* were used but this time with varying number of points i_{max} and j_{max}. The results are demonstrated in 5 and 6.

	$V(S_{init}, q_{init}, t = 0)$	Penalty	Gain
$i_{max} = 80, j_{max} = 60$	21.05445	0.84685	21.90129
$i_{max} = 160, j_{max} = 120$	21.23819	0.84243	22.08062
$i_{max} = 320, j_{max} = 240$	21.33229	0.83940	22.17168

Table 5: Solution for different number of points in the problem without full execution constraint (for *Parameterization 1*)

	$V(S_{init}, q_{init}, t = 0)$	Penalty	Gain
$i_{max} = 80, j_{max} = 60$	20.75707	0.38063	21.13770
$i_{max} = 160, j_{max} = 120$	21.00799	0.39555	21.40354
$i_{max} = 320, j_{max} = 240$	21.14804	0.40279	21.55082

Table 6: Solution for different number of points in the problem with full execution constraint (for Parameterization 1)

These results seem to confirm the previous guess that the failure of the solution is primarily linked to the chosen boundaries of the space Θ. It can be observed that for reasonable boundary values, even if the grid is discretized with a smaller number of points it still leads to roughly similar results. Moreover these results converge once the number of points is increased.

While comparing the results in terms of the full-execution constraint it can be noticed that in unconstrained problem the penalty for optimal strategy is higher but the overall gain is greater. This can be expected as for the *Parameterization 1* in unconstrained problem the terminal inventory is positive which leads to additional penalty. Nevertheless it is still optimal not to liquidate completely the position, which is confirmed by the higher gain. Figures 5 and 6 illustrate the comparison between the optimal strategies in the problem with and without the full execution constraint. In both cases the *Parameterization 1* and *Subdivision 2* are used.
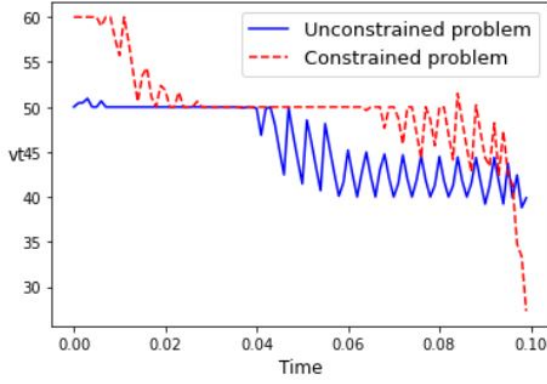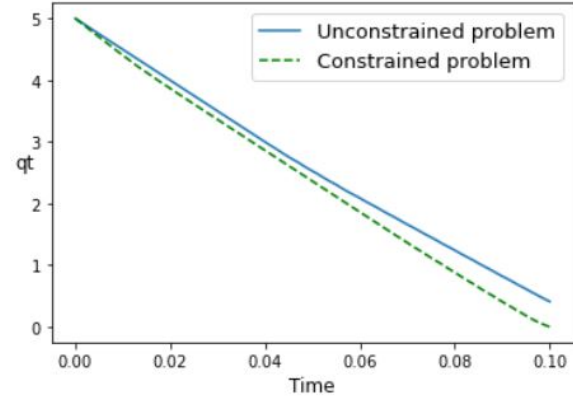
Figure 5: Optimal trading rates

Figure 6: Optimal inventories

It can be observed that in the full execution problem the optimal strategy is approximately selling at a constant rate. It has to be noted that the terminal decrease in trading rate is the discretization of time and is moreover linked to the fact that near the boundary q_{min} the set of admissible controls is restricted so that the inventory does not exceed the bounds as explained before. Concurrently, in unconstrained problem the trading rate seems to be slightly decreasing. Moreover there are clearly visible some irregular patterns in the behaviour of the trading rate which can indicate that there are some non-smooth changes of the value function V. The analysis of that issue suggests that reducing the spacing h of interval $[v_{min}, v_{max}]$ does not resolve it. For that reason it may be advisable to review the coefficients in the discretized HJB equation in order to prevent the oscillations even though there is no clear evidence that this is the source of the issue.

Also some convergence tests with respect to discretization of the time interval were conducted. The investigation of the result suggested that if the number of considered periods is too small then the problem of imprecise discretization occurs. On the other hand, the results are not converging with increasing the number of time points. It is most likely caused by the fact that, as mentioned before, the values near the boundary s_{max} grow extremely fast with each iteration influencing the values at other points on the grid and so increasing the number of time points leads to the spread of the regions in which the solution is false (and in general taking very high values). That is why in all the experiments the time discretization is fixed.

It also has to be mentioned that in the thesis only the solutions for fixed $q_{min} = 0$ are reported. This implies that even though it is possible to acquire assets a no short-selling constraint is imposed. The analysis showed that it is necessary in order to obtain reasonable solution, as without this constraint the optimal strategies in general tend to decrease assets up to the boundary q_{min}.

Optimal strategies for different model parameters

In this paragraph the different model parameterizations are studied and in particular their effects on the optimal strategies. Firstly, the *Parameterization 2* is considered in which the temporary market impact component is reduced. The results are illustrated in Figures 7 and 8 for constrained problem and in Figures 9 and 10 for unconstrained problem. It can be observed that reducing the temporary market impact component reduces the costs linked to trading, especially at higher rates, and hence the resulting liquidation strategies are more aggressive.
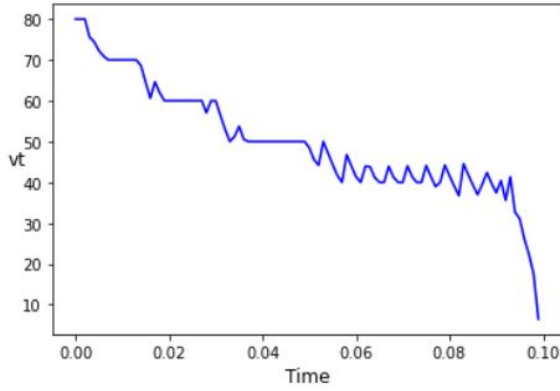
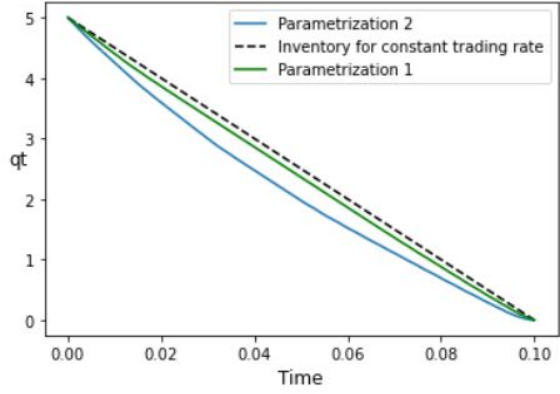Figure 7: Optimal trading rate in constrained problem for *Parameterization 2*
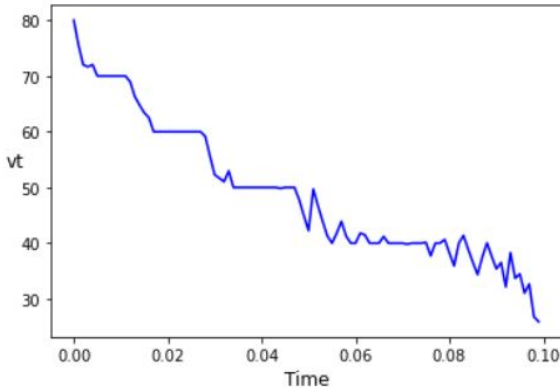

Figure 8: Optimal inventory

Figure 9: Optimal trading rate in unconstrained problem for *Parameterization 2*
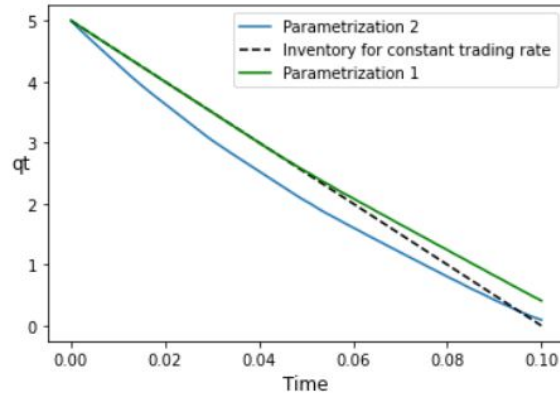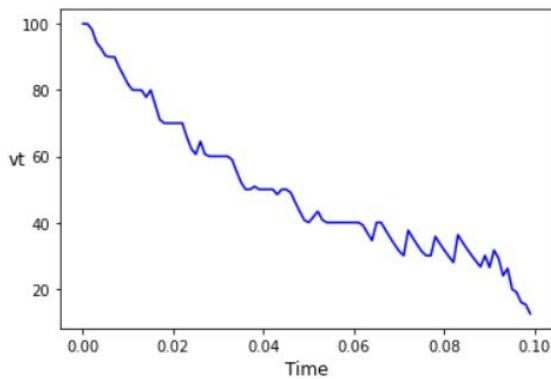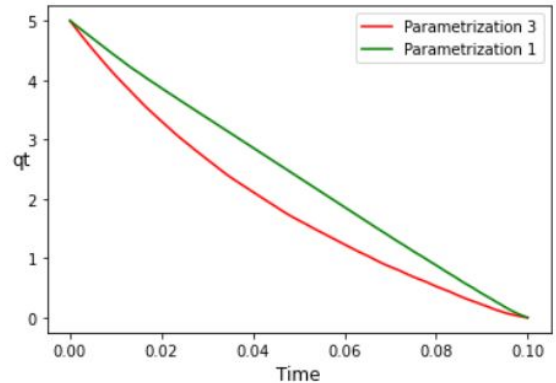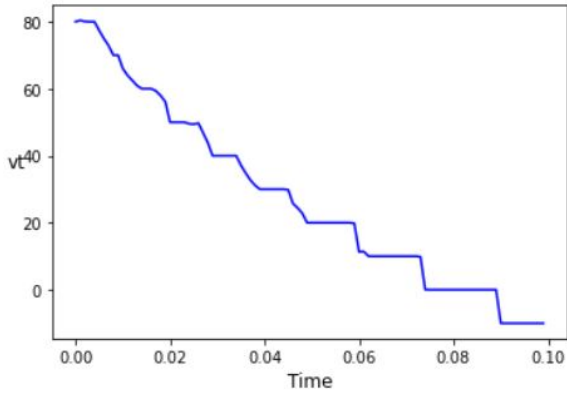

Figure 10: Optimal inventory

In the following experiment the *Parameterization 3* is considered. It is used as a study of the impact of penalization parameters on the model. Firstly, the increase of ϕ is expected to produce faster optimal strategies. Secondly, in the unconstrained problem the elimination of the penalty on terminal inventory is examined. The results are contained in the Figures 11 and 12 for the constrained problem and in Figures 13 and 14 for the unconstrained problem.

Figure 11: Optimal trading rate in constrained problem for *Parameterization 3*

Figure 12: Optimal inventory

34

Figure 13: Optimal trading rate in unconstrained problem for *Parameterization 2*
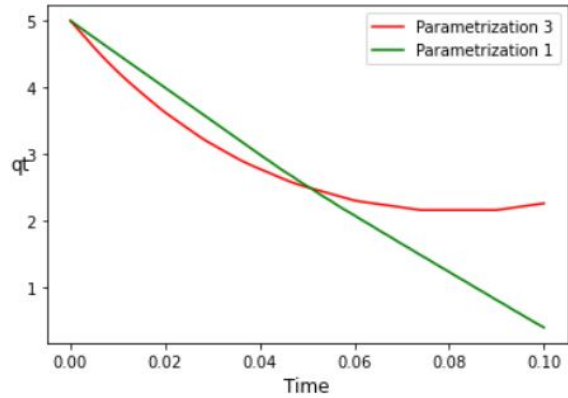

Figure 14: Optimal inventory

In the complete execution model higher penalty for remaining inventory led to faster liquidating optimal strategy, this effect seems to be more profound than in the case of reducing the trading costs related to temporary market impact component. Despite the increase of ϕ in unconstrained model the lack of penalization on terminal inventory led to leaving almost half of the initial portfolio unsold. Moreover it can be observed that near the terminal time according to that strategy investor should acquire some portion of assets. This example demonstrates that both forms of penalization are complementary and should be imposed at the same time to obtain efficient strategies. Additionally it may be reported that other experiments which are not reported in the thesis indicated that if the parameter ϕ is to small (for $\rho = 0$) then the optimal strategy may turn out to be a purely purchase program which ends up with terminal position higher that the initial one. In other words, the penalty it too small even to enforce the proper direction of the strategy.

It can be noticed that there is a similar pattern for all the cases that optimal trading rate is piecewise constant up to some oscillations. We can presume that this is caused by the discretization of the problem and the true strategy would be in fact smooth. This guess can be illustrated by comparing the results for different spacing h of the interval $[v_{min}, v_{max}]$. Figures 12 and 13 demonstrate such an example for unconstrained problem using *Parameterization 3*.
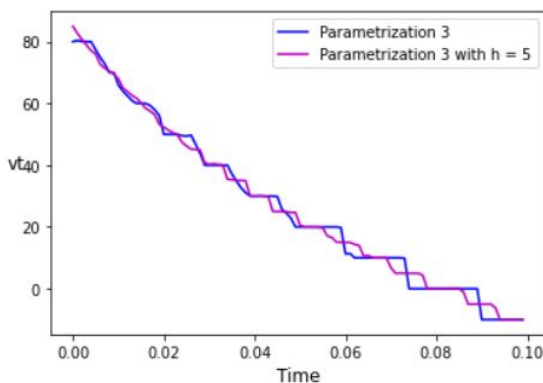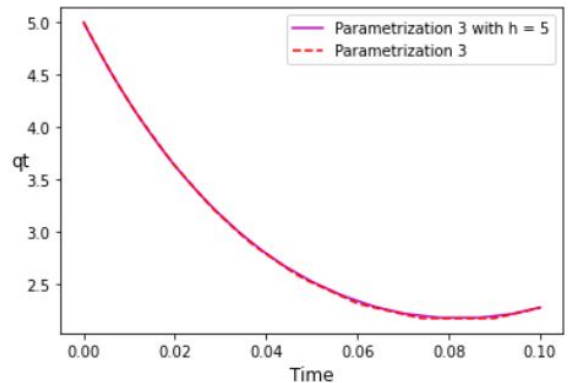

Figure 15: Trading rates for different h

Figure 16: Optimal strategies for different h

It can be clearly observed that even though the optimal inventory paths look very alike, the trading rate obtained with smaller spacing h is smoother. Firstly, it indicates that the solution seems to be converging for more precise discretizations of the interval of possible values attained by the trading rate. Secondly, it suggests that it may be effective to use smoothing techniques on obtained discrete approximations of optimal trading rate to obtain more precise curves (closer to a true solution).

In the subsequent example the *Parameterization 4* is considered. The aim of this experiment was to examine the effect of disregarding the penalty on the terminal inventory on the optimal strategies. Let's notice that without this penalty the optimization problem can be seen approximately as maximization of revenues which in additive market impact models results in the optimality of constant trading rate. It can be tested if the similar dependency is present in the considered case as well. The results are presented in Figures 17 and 18 for unconstrained problem and in Figures 19 and 20 for constrained problem. In the latter case it is quite clearly visible that the resulting strategy is indeed to sell at constant rate. The results for unconstrained problem are not that obvious due to excessive oscillations of the optimal trading rate. Moreover we observe that the average trading rate in that case is slightly lower than in complete execution model, which is not surprising.
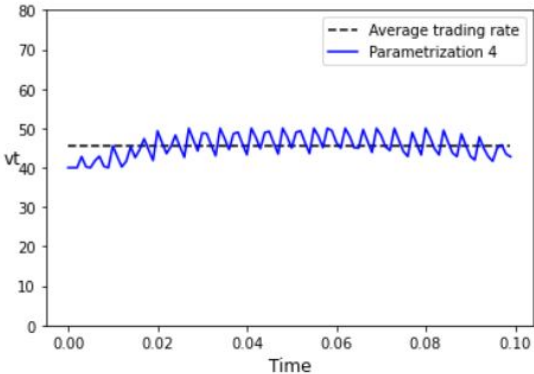

Figure 17: Optimal trading rate in unconstrained problem for *Parameterization 4*
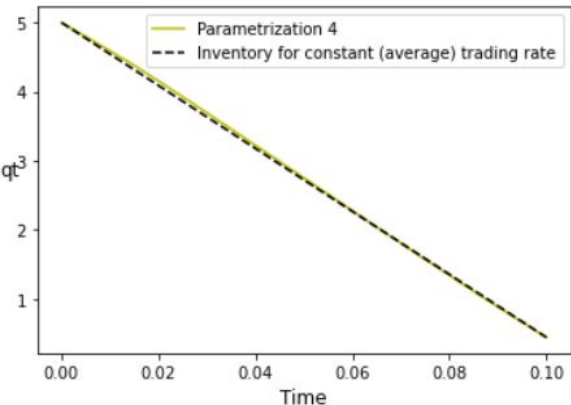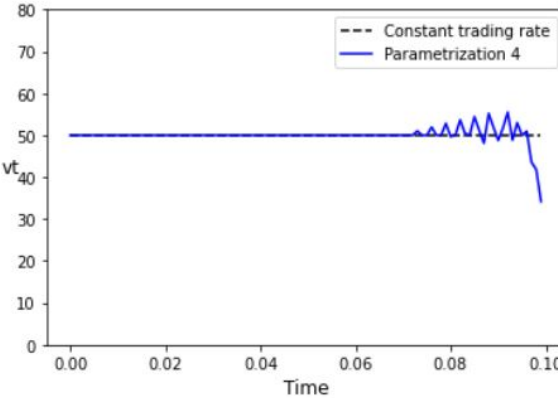

Figure 18: Optimal inventory

Figure 19: Optimal trading rate in constrained problem for *Parameterization 4*
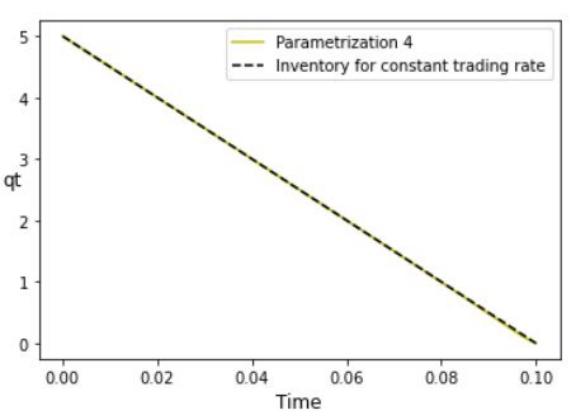

Figure 20: Optimal inventory

Last but not least, in Figures 21 and 22 the comparison between the numerical and analytical solutions is presented. It has to be kept in mind that these solutions correspond to different models with various assumptions about price diffusion and form of market impact. However it can be observed that the general shape of the curve is similar. Additionally, Figure 23 illustrates the magnitude of price impact contrasting the underlying, unaffected price process with the actual price affected by trading.
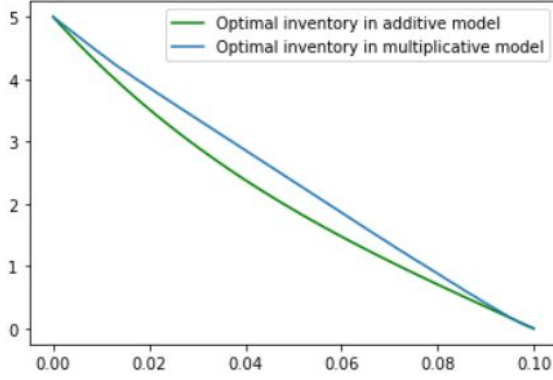
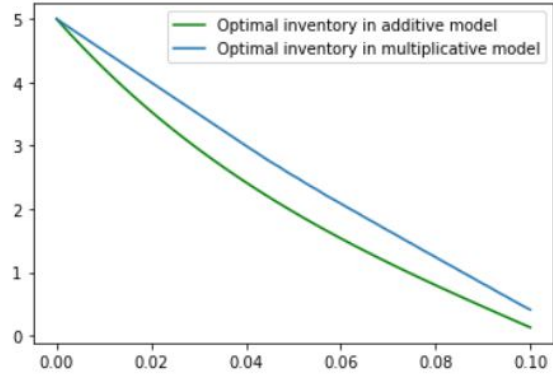Figure 21: Optimal inventories (in complete execution model)

Figure 22: Optimal inventories

Figure 23: Example of unaffected vs affected price trajectory

7 Conclusions

In this thesis an alternative formulation of optimal execution problem was described. We proposed to incorporate penalties on terminal and remaining inventory in the optimization criterion in order to enforce the resulting optimal strategies to match a typically expected behaviour. We argued that those penalties effectively act as a form of restriction of risk related to price fluctuation and even though they lack direct affiliation to classic risk measures, they can be easily used to compare different strategies in terms of riskiness. We considered both additive market impact model with prices following ABM dynamics and multiplicative impact framework with prices driven by GBM. In the former case we demonstrated different approaches to solving resulting problems analytically, although all of these methods relied on adjusting the optimization criterion to Linear-Quadratic optimal control framework. In the multiplicative market impact model, due to the lack of analytical solution, we used numerical discretization methods to solve the corresponding HJB equations. We then performed a set of experiments and tests which demonstrated that the obtained solution is stable under change of the model parameters and converges with the increase of discretization precision in s, q and v. We also found that the solution is strongly dependent on discretization of the time horizon. We reported that if the considered time points are too dense then the values become extremely large. We examined that this is caused by the imprecise assumption imposed as a boundary condition for s_{max}. As a result, values computed in regions close to that boundary were far from rational. Moreover, those error regions seem to be spreading with each iteration over time. To address this issue it may be advised to either scale

the considered space correspondingly to the number of time periods, or intervene into the method of calculating the values at the boundary. Furthermore, we observed certain oscillating behaviour of the optimal trading rates which may suggest that the method of discretizing the equations should be reviewed in terms of coefficients which may cause fluctuations. We also noted that the solution does not seem to work properly when short-selling is allowed. All of these issues raise a question about the limits of validity of the obtained solution. On the other hand, the results for numerous examples reported in the thesis seem to be rational, and convergence tests also indicate that in fact we managed to solve the posed problems.

A   Appendix

A.1   Solution of the Riccati equation

We have that

$$
\begin{cases}
0 = f'(t) - \phi + \frac{f(t)^2}{\eta} \\
f(T) = (\frac{\gamma}{2} - \rho)
\end{cases}
\tag{A.1}
$$

where the coefficients $-\phi$ and $\frac{1}{\eta}$ are constants thus the equation can be reduced to separable differential equation.

$$
\frac{f'(t)}{f(t)^2 - \phi\eta} = -\frac{1}{\eta}
\tag{A.2}
$$

using partial fraction decomposition

$$
\left(\frac{f'(t)}{f(t) - \sqrt{\phi\eta}} - \frac{f'(t)}{f(t) + \sqrt{\phi\eta}} \right) \frac{1}{2\sqrt{\phi\eta}} = -\frac{1}{\eta}
\tag{A.3}
$$

which is equivalent to

$$
\left(\frac{df(t)}{f(t) - \sqrt{\phi\eta}} - \frac{fd(t)}{f(t) + \sqrt{\phi\eta}} \right) \frac{1}{2\sqrt{\phi\eta}} = -2\sqrt{\frac{\phi}{\eta}} dt
\tag{A.4}
$$

Applying the integral operator

$$
\int_t^T \frac{df(s)}{f(s) - \sqrt{\phi\eta}} - \int_t^T \frac{df(s)}{f(s) + \sqrt{\phi\eta}}) = -\int_t^T 2\sqrt{\frac{\phi}{\eta}} ds
\tag{A.5}
$$

which together with the condition $f(T) = (\frac{\gamma}{2} - \rho)$ gives

$$
\ln\left(\underbrace{\left| \frac{\frac{\gamma}{2} - \rho - \sqrt{\frac{\phi}{\eta}}}{\frac{\gamma}{2} - \rho + \sqrt{\frac{\phi}{\eta}}} \right|}_{|M|} \right) - \ln\left(\left| \frac{f(t) - \sqrt{\frac{\phi}{\eta}}}{f(t) + \sqrt{\frac{\phi}{\eta}}} \right| \right) = -2\sqrt{\frac{\phi}{\eta}}(T - t)
\tag{A.6}
$$

so that

$$
\left| \frac{f(t) - \sqrt{\frac{\phi}{\eta}}}{f(t) + \sqrt{\frac{\phi}{\eta}}} \right| = e^{2\sqrt{\frac{\phi}{\eta}}(T-t) + \ln|M|}
\tag{A.7}
$$

Let's assume that $f(t) - \sqrt{\frac{\phi}{\eta}} > 0$, after rearranging the terms

$$
f(t) = \frac{1 + e^{2\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}}{1 - e^{2\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}} \sqrt{\phi\eta}
\tag{A.8}
$$

38

Denote $\kappa_t = \sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|$ then the equation rewrites as

$$f(t) = \frac{e^{-\kappa_t} + e^{\kappa_t}}{e^{-\kappa_t} - e^{\kappa_t}}\sqrt{\phi\eta} = -\frac{e^{\kappa_t} + e^{-\kappa_t}}{e^{\kappa_t} - e^{-\kappa_t}}\sqrt{\phi\eta} = -\frac{\cosh(\kappa_t)}{\sinh(\kappa_t)}\sqrt{\phi\eta} = -\coth(\kappa_t)\sqrt{\phi\eta} \tag{A.9}$$

Finally

$$f(t) = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\phi\eta} \tag{A.10}$$

A.2 Solution of the homogeneous ODE

We have

$$\dot{q}_t = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\frac{\phi}{\eta}}q_t \tag{A.11}$$

with the initial condition q_0
By separation of variables

$$\frac{dq_t}{q_t} = -\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\sqrt{\frac{\phi}{\eta}}dt \tag{A.12}$$

Applying the integral operator

$$\int_0^t \frac{dq_s}{q_s} = -\int_0^t \coth\left(\sqrt{\frac{\phi}{\eta}}(T-s) + \frac{1}{2}\ln|M|\right)\sqrt{\frac{\phi}{\eta}}ds \tag{A.13}$$

which is equal to

$$\ln\left(\frac{q_t}{q_0}\right) = \ln\left(\frac{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|\right)}\right) \tag{A.14}$$

which gives the solution

$$q_t^* = q_0\frac{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|\right)} \tag{A.15}$$

A.3 Solution of the first non-homogeneous ODE

We have

$$\dot{k}_{12}(t) = -\frac{1}{\eta}k_{11}(t)k_{12}(t) - \frac{1}{2} \tag{A.16}$$

which we rewrite as

$$\dot{k}_{12}(t) + p(t)k_{12}(t) = -\frac{1}{2} \tag{A.17}$$

where

$$p(t) = \frac{1}{\eta}k_{11}(t) = -\sqrt{\frac{\phi}{\eta}}\coth\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right) \tag{A.18}$$

Basic analysis of this non-homogeneous ODE indicates that the general solution is of the form

$$k_{12}(t) = s(t) + g(t) \tag{A.19}$$

where $s(t)$ is a particular solution to non-homogeneous ODE i.e. $\dot{s}(t) + p(t)s(t) = -\frac{1}{2}$ and $g(t)$ is the general solution to the corresponding homogeneous ODE

$$\dot{g}(t) + p(t)g(t) = 0 \tag{A.20}$$

that is

$$g(t) = e^{\int -p(t)dt} = \frac{A}{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)} = Ah(t) \tag{A.21}$$

for some suitable constant A.

Moreover we make an educated guess that function $s(t)$ is of the form

$$s(t) = w(t)h(t) \tag{A.22}$$

We then observe that

$$\begin{aligned}
\dot{s}(t) + p(t)s(t) &= w(t)\dot{h}(t) + \dot{w}(t)h(t) + p(t)w(t)h(t) \\
&= w(t)[\dot{h}(t) + p(t)h(t)] + \dot{w}(t)h(t)
\end{aligned} \tag{A.23}$$

and since $h(t)$ is itself a solution to homogeneous ODE ($\dot{h}(t) + p(t)h(t) = 0$) we have that

$$\dot{s}(t) + p(t)s(t) = \dot{w}(t)h(t) \tag{A.24}$$

substituting into ODE we obtain that

$$\dot{w}(t)h(t) = -\frac{1}{2} \tag{A.25}$$

and hence

$$\dot{w}(t) = -\frac{1}{2}\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right) \tag{A.26}$$

so

$$w(t) = \frac{1}{2}\sqrt{\frac{\eta}{\phi}}\left(\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\right) + B \tag{A.27}$$

for some constant B

Substituting into we get that

$$\begin{aligned}
k_{12}(t) = s(t) + g(t) &= w(t)h(t) + Ah(t) \\
&= h(t)\left(\frac{1}{2}\sqrt{\frac{\eta}{\phi}}\left(\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)\right) + C\right) \\
&= \frac{\sqrt{\eta}\cosh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}{2\sqrt{\phi}\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)} + \frac{C}{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)}
\end{aligned} \tag{A.28}$$

To find a constant C we use the terminal condition $k_{12}(T) = 0$ hence

$$C = -\frac{\sqrt{\eta}}{2\sqrt{\phi}}\cosh\left(\frac{1}{2}\ln|M|\right) \tag{A.29}$$

Finally

$$k_{12}(t) = \frac{\sqrt{\eta}}{2\sqrt{\phi}} \left(\coth \left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M| \right) - \frac{\cosh\left(\frac{1}{2}\ln|M|\right)}{\sinh\left(\sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|\right)} \right) \tag{A.30}$$

A.4 Solution of the second non-homogeneous ODE

We denote $\kappa_t = \sqrt{\frac{\phi}{\eta}}(T-t) + \frac{1}{2}\ln|M|$ so the ODE is

$$\dot{q}_t = -\coth\left(\kappa_t\right)\sqrt{\frac{\phi}{\eta}}q_t + \frac{\mu}{2\sqrt{\eta\phi}}\left(\coth\left(\kappa_t\right) - \frac{\cosh\left(\frac{1}{2}\ln|M|\right)}{\sinh\left(\kappa_t\right)}\right) \tag{A.31}$$

Similarly as before we rewrite it as

$$\dot{q}_t + p(t)q_t = f(t) \tag{A.32}$$

and assume that the general solution is of the form

$$q_t = s(t) + e^{\int -p(\kappa_t)dt} = s(t) + Ah(t) \tag{A.33}$$

with $h(t) = \sinh\left(\kappa_t\right)$

and that a particular solution $s(t)$ is of the form $s(t) = w(t)h(t)$
According to that guess and the fact that $\dot{h}(t) + p(t)h(t) = 0$ the ODE

$$\dot{s}_t + p(t)s_t = f(t) \tag{A.34}$$

can be rewritten as

$$\dot{w}(t)h(t) = f(t) \tag{A.35}$$

so

$$\dot{w}(t) = \frac{\mu}{2\sqrt{\eta\phi}}\left(\frac{\cosh\left(\kappa_t\right)}{\sinh^2\left(\kappa_t\right)} - \frac{\cosh\left(\frac{1}{2}\ln|M|\right)}{\sinh^2\left(\kappa_t\right)}\right) \tag{A.36}$$

and

$$w(t) = -\frac{\mu}{2\phi}\left(\int\frac{\cosh\left(\kappa_t\right)}{\sinh^2\left(\kappa_t\right)}d\kappa_t - \cosh\left(\frac{1}{2}\ln|M|\right)\int\frac{1}{\sinh^2\left(\kappa_t\right)}d\kappa_t\right) \tag{A.37}$$

in the first integral we use the substitution $u = \sinh\left(\kappa_t\right) \implies du = \cosh\left(\kappa_t\right)d\kappa_t$ which gives

$$w(t) = -\frac{\mu}{2\phi}\left(\int\frac{1}{u^2}du - \cosh\left(\frac{1}{2}\ln|M|\right)\int\text{csch}^2(\kappa_t)d\kappa_t\right) \tag{A.38}$$

which gives

$$w(t) = \frac{\mu}{2\phi}\left(\frac{1}{\sinh\left(\kappa_t\right)} - \cosh\left(\frac{1}{2}\ln|M|\right)\coth\left(\kappa_t\right)\right) + B \tag{A.39}$$

for suitable constant B. Substituting into guess

41

$$q_t = h(t)(w(t) + A) = \frac{\mu}{2\phi}\left(1 - \cosh\left(\frac{1}{2}\ln|M|\right)\cosh(\kappa_t)\right) + \sinh(\kappa_t)C \tag{A.40}$$

and in order to find the constant C we substitute for κ_t and write the equation for $t = 0$

$$q_0 = \frac{\mu}{2\phi}\left(1 - \cosh\left(\frac{1}{2}\ln|M|\right)\cosh(\beta)\right) + \sinh(\beta)C \tag{A.41}$$

where $\beta = \sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|$

$$\implies C = \frac{1}{\sinh(\beta)}\left(q_0 - \frac{\mu}{2\phi} + \frac{\mu}{2\phi}\cosh\left(\frac{1}{2}\ln|M|\right)\cosh(\beta)\right) \tag{A.42}$$

thus we obtain

$$q_t = \left(q_0 - \frac{\mu}{2\phi}\right)\frac{\sinh(\kappa_t)}{\sinh(\beta)} + \frac{\mu}{2\phi} + \frac{\mu}{2\phi}\left(\frac{\sinh(\kappa_t)\cosh(\beta)\cosh\left(\frac{1}{2}\ln|M|\right)}{\sinh(\beta)} - \cosh(\kappa_t)\cosh\left(\frac{1}{2}\ln|M|\right)\right) \tag{A.43}$$

which can be reformulated as

$$q_t = \left(q_0 - \frac{\mu}{2\phi}\right)\frac{\sinh(\kappa_t)}{\sinh(\beta)} + \frac{\mu}{2\phi} + \frac{\mu}{2\phi}\cosh\left(\frac{1}{2}\ln|M|\right)\left(\frac{\sinh(\kappa_t)\cosh(\beta) - \cosh(\kappa_t)\sinh(\beta)}{\sinh(\beta)}\right) \tag{A.44}$$

observe that

$$
\begin{aligned}
\frac{\sinh(\kappa_t)\cosh(\beta) - \cosh(\kappa_t)\sinh(\beta)}{\sinh(\beta)} &= \frac{\frac{1}{4}(e^{\kappa_t} - e^{-\kappa_t})(e^{\beta} + e^{-\beta}) - \frac{1}{4}(e^{\kappa_t} + e^{-\kappa_t})(e^{\beta} - e^{-\beta})}{\sinh(\beta)} \\
&= \frac{\frac{1}{4}(e^{\kappa_t+\beta} + e^{\kappa_t-\beta} - e^{-\kappa_t+\beta} - e^{-\kappa_t-\beta}) - \frac{1}{4}(e^{\kappa_t+\beta} - e^{\kappa_t-\beta} + e^{-\kappa_t+\beta} - e^{-\kappa_t-\beta})}{\sinh(\beta)} \\
&= \frac{\frac{1}{2}(e^{\kappa_t-\beta} - e^{-(\kappa_t-\beta)})}{\sinh(\beta)} = \frac{\sinh(\kappa_t - \beta)}{\sinh(\beta)} \\
&= \frac{\sinh\left(-\sqrt{\frac{\phi}{\eta}}t\right)}{\sinh(\beta)}
\end{aligned} \tag{A.45}
$$

with the last equality true because $\kappa_t - \beta = -\sqrt{\frac{\phi}{\eta}}t$
Finally

$$q_t = \left(q_0 - \frac{\mu}{2\phi}\right)\frac{\sinh(\kappa_t)}{\sinh(\beta)} + \frac{\mu}{2\phi} + \frac{\mu}{2\phi}\cosh\left(\frac{1}{2}\ln|M|\right)\left(\frac{\sinh\left(-\sqrt{\frac{\phi}{\eta}}t\right)}{\sinh(\beta)}\right) \tag{A.46}$$

where $\kappa_t = \sqrt{\frac{\phi}{\eta}}(T - t) + \frac{1}{2}\ln|M|$, $\beta = \sqrt{\frac{\phi}{\eta}}T + \frac{1}{2}\ln|M|$

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